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Principal-Agent Problem with Continuum of Constraints: The Infinite Dimensional Approach

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**Existence of Time-Invariant Settlements
in FEDWIRE-Like Payment Systems**

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The Principal-Agent Model with a Continuum of Constraints: The Infinite-Dimensional Approach

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ABSTRACT: This paper is a survey about the existence of optimal contract mechanisms in the principal-agent problems with moral hazard and adverse selection under a continuum of individual rationality and incentive compatibility constraints. It presents the main existing results under a unified notation, shows explicitly in what sense they are particular cases of the general model presented in Balder (JET, 68 (1996): 133-148), and provides a simple method (the model machine) to construct new models within the infinite dimensional approach.

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Introduction

When a principal hires an agent to manage a firm, she might face two problems: the problem of hidden action and the problem of hidden information. The hidden action appears because the principal can not observe directly what action is the agent taking. This problem is known in the literature as moral hazard. The hidden information in turn appears because of the informational asymmetries that might exist between the principal and the agent concerning the firm's opportunities.

The principal-agent problem is defined as the problem of designing a contract scheme that maximizes the principal's utility subject to two constraints: the individual rationality and the incentive compatibility. The first constraint induces the agent to accept the contract. This happens whenever the contract makes the agent achieve a minimum level of utility, the reservation utility level. In the case of moral hazard only, the second

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constraint induces the agent to take the action required by the principal. In the case of adverse selection, the second constraint induces the agent to report truthfully her actual type.

As pointed out by Hart & Holmström (1987), much of the theoretical results from studying principal-agent models with moral hazard have been obtained by assuming that the agent can take only two possible actions. When the agent is allowed to choose from a finite action space with more than two actions, many complications related to implementation, existence, and optimality come out [Mas-Collel, Whinston & Green (1995), p.503]. When the action space is a continuum, then matters are still more difficult, since the principal-agent problem turns into an optimization programming with a continuum of constraints. The introduction of adverse selection with a continuum of types simply doubles the difficulty, though the mathematical problem is the same in nature.

The most known method to cope with the continuum of constraints in the principal-agent model is the first-order approach. It consists in substituting the continuum of individual rationality and incentive compatibility constraints by a first-order condition for maximization. However, Mirlees (1975) pointed out that the first-order approach carries many problems itself, in the sense that it yields to local maxima, but may pick out some compensation scheme that does not imply the conditions for a global maximum. Grossman & Hart (1983) presented an alternative procedure to overcome the problems of the first-order approach pointed out by Mirlees (1975). They showed that if the agent's preferences over income lotteries are independent of action, then an action can be implemented by solving a standard convex programming problem. In addition, assuming the monotone likelihood condition and the convexity of the distribution function, they showed that the optimal contract is nondecreasing in output, provided the principal is risk-neutral [Grossman & Hart (1983) and Rogerson (1985)]. Then one difficulty of the first-order approach is therefore how to go from local maxima to global maximum without assuming too many conditions on the behavior of the players.

As argued above, under a continuum of constraints (or constraints that hold almost everywhere), the principal's maximization problem of designing optimal contracts is not an easy target to get. The most common method to deal with this problem, the first order approach [Grossman & Hart (1983) and Rogerson (1985)], does not provide in general equivalent solutions. In addition, many unnecessary assumptions on differentiability are required. In Grossman & Hart (1983) and Rogerson (1985), the existence problem is addressed by assuming only finitely many outcomes. Another important feature of the models in Grossman & Hart (1983) and Rogerson (1985) is that both the principal and the agent have the same probability beliefs and these beliefs are such that each of the finitely many outcomes has a positive probability regardless of the agent's action.

The best method to solve the problem is to face it exactly how it is: an infinite-dimensional problem. Thus the main tool is the infinite-dimensional analysis. *Grosso modo*, this approach focuses on modeling the set of available contracts as a subset of a infinite-dimensional space. I call this the *infinite-dimensional approach* or the infinite-dimensional principal-agent model. The strategy consists in setting conditions implying compactness

and nonemptiness of the set of all incentive compatible and individually rational contract selection mechanisms (i.e., the menu of contracts), and implying continuity of the principal's expected (over types and/or over states of the world) utility, so that a solution will always exist by the Weierstrass theorem. Notice however that compactness and continuity in infinite-dimensional spaces are more subtle and difficult than in finite-dimensional spaces.

Infinite-dimensional spaces arise naturally in many important and relevant economic situations. For instance, suppose that a certain contract pays off affinely according to the realization of a normally distributed random variable. Hence the payoff function of the contract is itself a normally distributed random variable. Suppose in addition that the beliefs about the events of the world are described by a probability measure that is absolutely continuous with respect to the Lebesgue measure and that its Radon-Nykodim derivative (i.e., its density function) is such a normal density. If we define two contracts as equivalent contracts whenever they generate the same payoff function almost everywhere, then we can describe the set of available contracts as embedded in the infinite-dimensional space of integrable functions with respect to a measure that is absolutely continuous with respect to the Lebesgue measure. So instead of working with a lot of functions that satisfy a lot of conditions, we can work directly with the infinite-dimensional space that describes the model.

Therefore the infinite-dimensional approach is not only a mathematical generalization. On the contrary it is a model that describes better most of the important economic situations. This is simply a more powerful point of view.

Page (1987) addressed for the first time the problem of formulating the principal-agent model in an infinite dimensional setting. By assuming that the contract space is infinite dimensional, he allowed the set of outcomes on which the contract payoffs are based to be uncountable, namely, a continuum. If we recall that Grossman & Hart (1983) and Rogerson (1985) treat the existence problem assuming only finitely many outcomes, we can easily see that Page (1987)'s model is more general. Moreover, in Page (1987) the principal and the agent are allowed to disagree on the probabilities of the events as functions of the agent's action. They can even disagree on events of probability zero as a result of particular actions. However, Page (1987) did not consider the problem of adverse selection. Indeed, he considered only the existence of optimal contracts under moral hazard when the action space is a continuum.

Things changed after Balder (1989 and 1990) extended to infinite dimensional spaces a well-known theorem of Komlós. Specifically, this is a theorem that characterizes sequential compactness by taking subsequences that converge in average, which is a natural counterpart of the strong law of large numbers. Despite it is a mathematical result, it was this theorem that allowed the subsequent development of a research agenda applying infinite dimensional analysis to the principal-agent problem under moral hazard and adverse selection with a continuum of constraints. All papers in this research agenda use Balder (1989 and 1990)'s result, with no exception.

In order to understand the importance of Balder (1989 and 1990)'s result to this literature, we have to make a brief digression. We know from the finite dimensional optimization theories that if the constraint set is nonempty, convex, and compact, and the objective function is continuous, then a solution exists and, by Berge's maximum theorem, that the value function is well behaved. A moment of reflection makes us ask whether this is so in infinite dimensional spaces. Problems concerning nonemptiness of the interior of positive cones, weak compactness of the closure of the unit ball, etc., are much more complicated. These problems are not addressed here, but it is worth mentioning that Balder (1989 and 1990)'s result provides a method to show that a subset of an infinite dimensional space is relatively compact. Specifically, by using the generalized Komlós theorem, the difficult problem of finding a suitable topology for the set of contract selection mechanisms can be reduced to the easier problem of identifying topological properties for the contract set that guarantee the existence of an optimal contract mechanism [Page (1991), p. 323-324]. To motivate the reader, in some principal-agent models the menu of feasible contracts is the set of probability measures on a suitable space of functions. Such a space of functions describes the type of payoff functions that players face (e.g., increasing functions, continuous functions, bounded functions, etc.). These are exactly the models surveyed here.

The infinite dimensional approach has been applied to many different economic problems. For example screening problems with moral hazard [Page (1992a)], optimal auctions with many agents [Page (1998)], Stackelberg games with incomplete information [Page (1989)], Bayesian incentive compatible mechanisms [Page (1992b)], existence of Nash equilibria [Balder (1995)], and the principal-agent problem [Page (1991 and 1994), Balder & Yannelis (1993), and Balder (1996)].

In this paper I present a survey on the existence problem in the principal-agent models with moral hazard and adverse selection within the infinite dimensional approach. I define the infinite dimensional approach to the principal-agent model with moral hazard and adverse selection as the model described in Balder (1996). For instance, it can be realized that all other models surveyed here are indeed special cases of Balder (1996)'s. In section 1, I present the infinite dimensional approach as an infinite dimensional programming problem and show that such a problem is well posed. Also, in this section I introduce the notation that is going to be used throughout the survey, so that comparisons can be easy to the reader. Sometimes I call this model the general model. In section 2 the existence results are presented. Balder (1996) distinguishes two kinds of individual rationality and incentive compatibility constraints: those that hold everywhere and those that hold almost everywhere (i.e., except for a set of measure zero). Then there are two kinds of maximization the principal might solve. For each problem a theorem of existence is provided. Balder (1996) also introduces the new concept of misspecification correspondence. Such a correspondence describes the subset of types that can be misreported by the agent. This correspondence has proved to be useful in modeling the multi-agent set-up. Section 3 presents the model of state-contingent contracts as in Balder & Yannelis (1993). In this model the set of potential payoff functions is contingent to the state of the world. In this section I show how this model is indeed a special case of Balder

(1996)'s. In order to do that I show that all the assumptions in Balder & Yannelis (1993) imply the assumptions of the general model. Section 4 presents the model of mixed contracts as in Page (1994). By mixed contracts Page (1994) understands probability measures on the space of payoff functions under which the contract mechanism is going to be selected. This method is known also as direct probabilistic mechanism selection and its origin can be found in Myerson (1982). This is also a principal-multi-agent model. Section 5 presents Page (1991)'s model of one principal, one agent and mixed contracts. Obviously this is a special case of Page (1994)'s model. In all these sections I show explicitly how to encapsulate the models in the general model. Balder (1996) only states briefly that his model is more general. This survey makes this statement clear, in the sense that I show deeply that every assumption of the general model are satisfied by each particular model. When the proof of a result is reproduced from the original paper, the author is referred to in the statement of the proposition. However some details of the original proofs are worked out a bit. In section 6 I present some spaces of payoff functions that generate menus of contracts that satisfy the assumptions of the general model. I realized that the papers in this research agenda follow a procedure to construct the models. I present this algorithm in section 7 and call it the model machine. Finally, section 8 concludes the survey and presents a synoptic table. It includes the models surveyed here and some other models not surveyed.

Because of the very nature of the existence problem, this survey is pretty technical. Issues like efficiency, second-best solution, etc., are not addressed here. Nevertheless, it summarizes the essential tool necessary to formulate principal-agent problems under continua. Moreover, it presents the main results on existence in this agenda with a unified notation, and shows explicitly (see the model machine in section 7) in what sense the existing models follow the same method of construction.

1. The infinite-dimensional approach

1.1 Introduction

In this section I present the principal-agent model as given in Balder (1996), which includes most of the results on the existence of optimal contracts with a *continuum* of incentive-compatibility and individual rationality constraints as particular cases. His model is general enough to treat simultaneously moral hazard and adverse selection problems.

The main contribution of Balder (1996)'s paper was to catch the topological assumptions on the feasible set for the principal, i.e., the menu of contracts, that underlie all the existing papers using the infinite-dimensional approach together with Komlos sequential compactness.

In the problem with moral hazard only, the principal is fully informed about the agent's type. Hence the problem of getting an optimal contract arises because the principal can not control the agent's action via contracts contingent on the agent's action. However,

as argued in the introduction, this survey concerns to models that consider also adverse selection.

In the case the principal is uncertain about the agent's type, there is a problem of adverse selection. If the principal is supposed to own beliefs concerning agent's types, the problem of designing optimal contracts has to take these beliefs into account.

The rest of the section presents Balder (1996)'s model in a more synthetic way. By this I mean that I simplified notation a bit and condensed his nine assumptions into four. All the economic entities and respective notations introduced in this section are kept the same throughout this survey. This makes comparisons easier to the reader. After describing the assumptions of the model (section 1.2), it is proved that the set of constraints is nonempty (proposition 1). Though not important for the main results of the model, I prove (proposition 2) that both kinds of feasible sets defined in Balder (1996) are convex. Actually this is a straightforward conclusion. The only merit of this result is to show that the feasible sets are much well behaved than they appear at a first sight (section 1.3).

1.2. *The set-up*

The structure of the agent's types is described by the probability space (T, \mathfrak{T}, ν) , where \mathfrak{T} is a σ -field on T and ν is an atomless measure. The continuum feature of this space is caught by the assumption that the probability measure on types is atomless. Then there does not exist any singleton type with positive probability measure. This is not very restrictive at all, because the agenda of the infinite-dimensional approach concerns exactly with continua of economic entities involved in the model. On the other hand, it does not mean that continuum of types is more real than finiteness of types. It just models different economic situations that are more reasonable to be thought as continua. When the principal faces the incentive compatibility and the individual rationality constraints on the agent's utility, it is this continuum that makes her maximization problem infinite-dimensional. The same could happen if the continuum were imposed on, e.g., possible effort levels available to the agent. The analysis is quite the same. The usefulness of the infinite dimensional approach rises because under continua it is the best approach.

In order to describe the set of contracts available to the principal within an infinite-dimensional framework, let (E, τ) be a Hausdorff locally convex topological vector space endowed with a topology τ . The menu of contracts is given by a subset $K \subset E$ satisfying the following assumption:

(A1) $K \subset E$ is convex, compact, and metrizable for the relative topology inherited from E .

So a contract is an element $x \in K$. Endow K with the Borel- σ -field \mathfrak{N} generated by the relative topology on K . Any $(\mathfrak{T}, \mathfrak{N})$ -measurable function $f: T \rightarrow K$ is said to be a *contract selection mechanism*. Hence once the principal adopts the contract selection mechanism f , she

has to offer the contract $f(t)$ to the agent provided the agent reports type t . This assumption is crucial for the regularity of the principal's problem. It eliminates nonexistence problems and guarantees the continuity of the expected utilities of the players [Page (1991), p. 328-329]. These topological and dimensionality assumptions on the menu of contracts are general enough to include all of the most common infinite-dimensional contract spaces. Indeed, some of the particular cases of this infinite-dimensional approach describe the menu of contracts as the set of integrable selections of some suitable Banach-space valued correspondence [Balder & Yannelis (1993)]. Other applications consider the set of probability measures on compacta [Page (1989, 1991, 1994, and 1998)]. The infinite-dimensionality of the menu of contracts is not just a mathematical generalization. It has the virtue of being the right space of contracts to model most of economic environments involving continua of some underlying entity. In section 6 some infinite dimensional spaces of payoff functions that generate menus of contracts that satisfy assumption (A1) are provided.

The moral hazard problem with a continuum of actions is incorporated in the set K . Indeed, as we will see later in this section, the actions taken by the agent from the set K influence the principal's utility.

The set of rational contracts for the type- t agent has to be a subset of the available menu of contracts. This is described by the *rational contract correspondence* $\Gamma: T \rightarrow \mathcal{P}(K)$ (where $\mathcal{P}(K)$ is the collection of all nonempty subsets of K) satisfying the following assumption:

(A2) $\forall t \in T, \Gamma(t)$ is convex and (relatively) closed.

Note that the only assumption on the rational contract correspondence is that it is convex-closed valued. No measurability assumption is imposed.

Each type- t agent is allowed to misreport her type to the principal. This is described by the *misspecification correspondence* $M: T \rightarrow \mathcal{P}(T)$. Again, no measurability is imposed on M . Such a misspecification correspondence is a new concept and as will be seen later, it allows for the simultaneous treatment of single-agents and multi-agents models. In general, the assumption $M(t) = T$ corresponds to the most common existing hypothesis, that is, the agent is allowed to misreport, whatever her type. However nontrivial misspecification correspondences can be used. This concept appeared by the first time in Balder (1996). Though all models surveyed in this paper assume somehow trivial misspecification correspondences, it is clear by the very definition of the misspecification correspondence that this concept is able to enrich the existing models by allowing exogenous constraints on misreporting or limited willingness to misreport.

The agent's utility function is given by the $\mathfrak{J} \otimes \mathfrak{N}$ -measurable function $U: T \times K \rightarrow \mathcal{R}$ satisfying the following assumption:

(A3) $\forall t \in T, U(t, \cdot): \Gamma(t) \rightarrow \mathcal{R}$ is continuous on K and affine on $\Gamma(t)$.

This assumption guarantees the existence of an optimal action for the agent, whatever her type.

Two kinds of individual rationality are considered as well as two kinds of incentive compatibility. They differ essentially by their validity either almost surely or everywhere. Balder (1996) captured very well this feature in the infinite dimensional agenda, so that this is an important cut-off aspect that distinguishes many models.

DEFINITION: A contract selection mechanism $f: T \rightarrow K$ is *almost surely individually rational* (asIR) if $f(t) \in \Gamma(t)$, ν -a.s., i.e., if f is a measurable selection of the correspondence Γ . Let $S(\text{asIR})$ be the set of all asIR contract selection mechanisms. A contract selection mechanism $f: T \rightarrow K$ is *individually rational* (IR) if $f(t) \in \Gamma(t)$, $\forall t \in T$. Let $S(\text{IR})$ be the set of all IR contract selection mechanisms.

DEFINITION: A contract selection mechanism $f: T \rightarrow K$ is *almost surely incentive compatible* (asIC) if $U(t, f(t)) \geq U(t, f(t'))$, $\forall t' \in M(t)$, ν -a.s. Let $S(\text{asIC})$ be the set of all asIC contract selection mechanisms. A contract selection mechanism $f: T \rightarrow K$ is *incentive compatible* (IC) if $U(t, f(t)) \geq U(t, f(t'))$, $\forall t' \in M(t)$, $\forall t \in T$. Let $S(\text{IC})$ be the set of all IC contract selection mechanisms.

Notice that the incentive compatibility is defined in terms of the misspecification correspondence. Then in the case that $M(t)$ is a proper subset of the type space T , the resulting incentive compatibility constraint is smaller than in the trivial case $M(t) = T$. Therefore, should this be the case, it is clear that the principal would get higher utility. This fact highlights the conjecture that the utility level of the second best could be improved upon, if efficiency were the issue. Since the only issue of this survey is the existence, I will not care about this conjecture here. Moreover, all paper surveyed here address only the question of existence.

The principal's utility function is given by the $\mathfrak{I} \otimes \mathfrak{N}$ -measurable function $V: T \times K \rightarrow [-\infty, \infty)$ satisfying the following assumption:

(A4) (i) $\forall f \in S(\text{asIR}) \cap S(\text{asIC})$, the function $\Psi: T \rightarrow [-\infty, \infty)$ defined by $\Psi(t) = V(t, f(t))$ is \mathfrak{I} -measurable. (ii) $\forall t \in T$, $V(t, \cdot): \Gamma(t) \rightarrow [-\infty, \infty)$ is concave and upper-semicontinuous on $\Gamma(t)$. (iii) V is integrably upper bounded on $S(\text{asIR}) \cap S(\text{asIC})$ with respect to T , i.e., $\exists \psi \in L^1(\nu)$ such that, $\forall f \in S(\text{asIR}) \cap S(\text{asIC})$, $V(t, f(t)) \leq \psi(t)$, ν -a.s., where $L^1(\nu)$ is the set of real-valued ν -integrable functions on T .

The role of assumption (A4-i) is technical. The measurability of the utility is a necessary condition to define a meaningful principal's expected utility over agent's types, that is, to deal with the adverse selection problem. Assumption (A4-ii) says that the principal's utility is well behaved. It means that for each reported type, the principal prefers to diversify individually rational contract selection mechanisms. In other words, the type-contingent upper contour set of the principal's preferences is convex and closed. Assumption (A4-iii) guarantees that the principal's expected utility over types is finite.

In general, the set of rational contracts for the agent is defined via some *reservation utility function* $r:T \rightarrow \mathcal{R}$, so that a contract $x \in K$ is individually rational for the type- t agent if $U(t,x) \geq r(t)$. In this case the rational contracts correspondence is clearly given by $\Gamma(t) = \{x \in K: U(t,x) \geq r(t)\}$. Since no measurability condition is imposed to Γ , no measurability condition is imposed to the reservation utility function r either.

1.3. The menus of contracts are well behaved

Now we will show that both kinds of menus of contracts are nonempty (proposition 1) and convex (proposition 2). These menus of contracts are given by $S(IR) \cap S(IC)$ and $S(asIR) \cap S(asIC)$. This shows that the set for the maximization program the principal must solve is in some sense well posed.

The following proposition assures that the sets of (a.s.) individually rational and (a.s.) incentive compatible contract selection mechanisms are non-empty.

PROPOSITION 1 (Balder, 1996): *Assume (A1)-(A4), and suppose that the rational contracts correspondence is defined via some reservation utility, i.e., $\Gamma(t) = \{x \in K: U(t,x) \geq r(t)\}$.*

(a) *If $\Gamma(t) \neq \emptyset$, v-a.s., then $S(asIR) \cap S(asIC) \neq \emptyset$.*

(b) *If $\Gamma(t) \neq \emptyset$, $\forall t \in T$, then $S(IR) \cap S(IC) \neq \emptyset$.*

PROOF: Define the correspondence $\Delta:T \rightarrow P(K)$ by $\Delta(t) = \operatorname{argmax}\{U(t,x): x \in K\}$. Since K is compact and metrizable, and $U(t,\cdot)$ is continuous on K , $\forall t \in T$, and is $\mathfrak{I} \otimes \mathfrak{N}$ -measurable, it follows that Δ is $(\mathfrak{I}, \operatorname{Bor}(P(K)))$ -measurable, where $\operatorname{Bor}(P(K))$ is the Borel- σ -field on $P(K)$ generated by the Hausdorff topology of closed convergence. Since U is continuous on compact K , it follows from Weierstrass theorem that $\Delta(t) \neq \emptyset$. Then by the measurable selection theorem, there exists a measurable selection $\varphi^*:T \rightarrow K$ of Δ , i.e., $\varphi^*(t) \in \Delta(t)$, $\forall t \in T$. Then $U(t,\varphi^*(t)) \geq \sup\{U(t,x): x \in K\}$. In particular, we have that $U(t,\varphi^*(t)) \geq U(t,\varphi^*(t'))$, $\forall t' \in M(t)$. This means that $\varphi^* \in S(IC)$.

It is easy to see that $S(IC) \subset S(asIC)$. Indeed, for any incentive compatible contract selection mechanism $g \in S(IC)$, we have that, $\forall t \in T$, $U(t,g(t)) \geq U(t,g(t'))$, $\forall t' \in T$. Since this holds $\forall t \in T$, it holds in particular v-a.s., hence $g \in S(asIC)$, so $S(IC) \subset S(asIC)$.

Therefore, $\varphi^* \in S(asIC)$ as well.

Now assume that $\Gamma(t) \neq \emptyset$, v-a.s. Since $U(t, \varphi^*(t)) \geq \sup\{U(t, x) : x \in K\}$, $\forall t \in T$, and since $U(t, x) \geq r(t)$, $\forall x \in \Gamma(t)$, v-a.s., we have that $U(t, \varphi^*(t)) \geq r(t)$, v-a.s., hence $\varphi^* \in S(asIR)$. Therefore, $\varphi^* \in S(asIR) \cap S(asIC)$.

Now assume that $\Gamma(t) \neq \emptyset$, $\forall t \in T$. By the same argument as above, we have that $U(t, \varphi^*(t)) \geq r(t)$, $\forall t \in T$, hence $\varphi^* \in S(IR)$. Therefore, $\varphi^* \in S(IR) \cap S(IC)$. ■

Since Balder (1996) allows for two different kinds of individual rationality and incentive compatibility, the feasible set for the principal can be either $S(asIR) \cap S(asIC)$ or $S(IR) \cap S(IC)$. I call $S(asIR) \cap S(asIC)$ the *quasi-feasible set (or the menu of quasi-feasible contracts)*, and $S(IR) \cap S(IC)$ the *feasible set (or the menu of feasible contracts)*. Then proposition 1 gives sufficient conditions under which the menus of quasi-feasible contracts and the menu of feasible contracts are nonempty.

Balder (1996) did not prove that both menus of quasi-feasible and feasible contracts are convex. Indeed, this is not important for the main results on existence, since compactness suffices. However convexity is interesting by itself, since it shows that the constraints are much well behaved than at the first sight. Furthermore, convexity of the feasible sets is important for many particular cases of the general model. For instance, in Balder & Yannelis (1993) (section 3) the menu of contracts is given by the set of integrable selections of the contingent potential contract payoffs correspondence, hence it is convex, provided the correspondence is convex. Therefore, I will prove in proposition 2 below that under the same conditions of proposition 1, the menus of quasi-feasible and feasible contracts are convex.

PROPOSITION 2: *Assume that (A1)-(A4) hold and suppose that the rational contracts correspondence is defined via some reservation utility, i.e., $\Gamma(t) = \{x \in K : U(t, x) \geq r(t)\}$.*

(a) *If $\Gamma(t) \neq \emptyset$, v-a.s., then $S(asIR) \cap S(asIC)$ is convex.*

(b) *If $\Gamma(t) \neq \emptyset$, $\forall t \in T$, then $S(IR) \cap S(IC)$ is convex.*

PROOF: (a) Let $\Gamma(t) \neq \emptyset$, v-a.s. By proposition 1, $S(asIR) \cap S(asIC) \neq \emptyset$. Then we can take arbitrary $f, g \in S(asIR) \cap S(asIC)$. Let $0 \leq \alpha \leq 1$. Consider $b = \alpha f + (1 - \alpha)g$.

I claim that b is well defined, i.e., that b takes values in K and is $(\mathfrak{F}, \mathfrak{N})$ -measurable. Indeed, since K is convex, the contract selection mechanism b satisfies, $\forall t \in T$, $b(t) = \alpha f(t) + (1 - \alpha)g(t) \in K$. Since f and g are both $(\mathfrak{F}, \mathfrak{N})$ -measurable, we have that b is $(\mathfrak{F}, \mathfrak{N})$ -measurable as well. Then the contract selection mechanism b is well defined.

Now I will prove that $S(asIR)$ is convex. Since f is asIR, there exists a \mathbf{v} -null set $N \in \mathfrak{S}$ (i.e., $\mathbf{v}(N) = 0$) such that $f(t) \in \Gamma(t)$, $\forall t \in T \setminus N$. Since g is asIR, there exists a \mathbf{v} -null set $N' \in \mathfrak{S}$ (i.e., $\mathbf{v}(N') = 0$) such that $f(t) \in \Gamma(t)$, $\forall t \in T \setminus N'$. Let $N^* = N \cup N'$. Clearly $N^* \in \mathfrak{S}$ and $\mathbf{v}(N^*) = 0$. Since by (A2), $\Gamma(t)$ is convex, $\forall t \in T$, it is convex in particular $\forall t \in T \setminus N^*$. Since $T \setminus N^* \subset T \setminus N$ and $T \setminus N^* \subset T \setminus N'$, it follows in particular that $f(t), g(t) \in \Gamma(t)$, $\forall t \in T \setminus N^*$, which implies that $b(t) = \alpha f(t) + (1 - \alpha)g(t) \in \Gamma(t)$, $\forall t \in T \setminus N^*$, i.e., \mathbf{v} -a.s. Then $b \in S(asIR)$. So, $S(asIR)$ is convex.

Now I will prove that $S(asIC)$ is convex. Since $f \in S(asIC)$, we have that there exists $N^o \in \mathfrak{S}$ with $\mathbf{v}(N^o) = 0$ such that $U(t, f(t)) \geq U(t, f(t'))$, $\forall t' \in M(t)$, $\forall t \in T \setminus N^o$. Since $g \in S(asIC)$, we have that there exists $N' \in \mathfrak{S}$ with $\mathbf{v}(N') = 0$ such that $U(t, g(t)) \geq U(t, g(t'))$, $\forall t' \in M(t)$, $\forall t \in T \setminus N'$. Let $N^{**} = N^o \cup N'$. Then clearly $N^{**} \in \mathfrak{T}$ with $\mathbf{v}(N^{**}) = 0$ and $\forall t \in T \setminus N^{**}$ and $\forall t' \in M(t)$: $U(t, f(t)) \geq U(t, f(t'))$ and $U(t, g(t)) \geq U(t, g(t'))$. Since by (A2) $U(t, \cdot)$ is affine on $\Gamma(t)$, we have $U(t, b(t)) = U(t, \alpha f(t) + (1 - \alpha)g(t)) = \alpha U(t, f(t)) + (1 - \alpha)U(t, g(t)) \geq \alpha U(t, f(t')) + (1 - \alpha)U(t, g(t')) = U(t, \alpha f(t') + (1 - \alpha)g(t')) = U(t, b(t'))$. Therefore: $U(t, b(t)) \geq U(t, b(t'))$, $\forall t \in T \setminus N^{**}$ (i.e., \mathbf{v} -a.s.) and $\forall t' \in M(t)$, i.e., $b \in S(asIC)$. So $S(asIC)$ is convex.

Then $S(asIR) \cap S(asIC)$ is convex.

(b) Let $\Gamma(t) \neq \emptyset$, $\forall t \in T$. By proposition 1, $S(IR) \cap S(IC) \neq \emptyset$. Then we can take arbitrary $f, g \in S(IR) \cap S(IC)$. Let $0 \leq \alpha \leq 1$. Consider $b = \alpha f + (1 - \alpha)g$.

I claim that b is well defined. Indeed, since K is convex, the contract selection mechanism satisfies, $\forall t \in T$, $b(t) = \alpha f(t) + (1 - \alpha)g(t) \in K$. Since f and g are both $(\mathfrak{S}, \mathfrak{N})$ -measurable, we have that b is $(\mathfrak{S}, \mathfrak{N})$ -measurable as well. Then the contract selection mechanism b is well defined.

First, I will show that $S(IR)$ is convex. Since f is IR, $f(t) \in \Gamma(t)$, $\forall t \in T$. Since g is IR, $f(t) \in \Gamma(t)$, $\forall t \in T$. Since by (A2), $\Gamma(t)$ is convex, $\forall t \in T$, it follows that $f(t), g(t) \in \Gamma(t)$, $\forall t \in T$, implies $b(t) = \alpha f(t) + (1 - \alpha)g(t) \in \Gamma(t)$, $\forall t \in T$. Then $b \in S(IR)$. So, $S(IR)$ is convex.

Finally, I will show that $S(IC)$ is convex. Since $f \in S(IC)$, we have that $U(t, f(t)) \geq U(t, f(t'))$, $\forall t' \in M(t)$, $\forall t \in T$. Since $g \in S(IC)$, we have that $U(t, g(t)) \geq U(t, g(t'))$, $\forall t' \in M(t)$, $\forall t \in T$. Then $\forall t \in T$ and $\forall t' \in M(t)$: $U(t, f(t)) \geq U(t, f(t'))$ and $U(t, g(t)) \geq U(t, g(t'))$. Since by (A2) $U(t, \cdot)$ is affine on $\Gamma(t)$, we have $U(t, b(t)) = U(t, \alpha f(t) + (1 - \alpha)g(t)) = \alpha U(t, f(t)) + (1 - \alpha)U(t, g(t)) \geq \alpha U(t, f(t')) + (1 - \alpha)U(t, g(t')) = U(t, \alpha f(t') + (1 - \alpha)g(t')) = U(t, b(t'))$. Therefore: $U(t, b(t)) \geq U(t, b(t'))$, $\forall t \in T$ and $\forall t' \in M(t)$, i.e., $b \in S(IC)$. So $S(IC)$ is convex.

Then $S(IR) \cap S(IC)$ is convex. ■

2. Existence of optimal contracts

2.1 Introduction

In this section I present the existence theorems for the infinite-dimensional principal-agent model. In Balder (1996) there are two existence theorems, depending on the feasibility

everywhere or almost everywhere. I named these existence results respectively *quasi-existence theorem* and *existence theorem*.

In what follows I pose two kinds of maximization programming the principal might face, depending on the kind of menu of contracts she is constrained to. I call the first of them *quasi-maximization problem* and the second one simply *maximization problem*. Afterwards, existence results for both problems are proved.

2.2 The principal's infinite dimensional programming

Given the incomplete information aspect of the general model, the principal is concerned with maximizing her expected utility over the set of types of the agent. As we already know, this is because of the adverse selection problem she faces concerned agent's type. Then for a given contract selection mechanism $f:T \rightarrow K$, the expected utility of the principal according to her subjective probability belief ν is defined by the functional:

$$I_V(f) = \int_T V(t, f(t)) \nu(dt)$$

The problem faced by the principal is that of finding a contract selection mechanism f that assigns for each type- t agent a contract $f(t)$ in such a way that the type- t agent is willing to participate and has incentive to honestly report her type to the principal. In other words, f has to be individually rational and incentive compatible. These constraints may hold almost everywhere or everywhere. Hence there are two distinct problems for the principal.

Define the principal's *quasi-maximization problem* as:

$$(asP) \quad \sup_{f \in \mathcal{S}_{(asIR)} \cap \mathcal{S}_{(asIC)}} I_V(f)$$

Similarly, define the principal's *maximization problem* as:

$$(P) \quad \sup_{f \in \mathcal{S}_{(IR)} \cap \mathcal{S}_{(IC)}} I_V(f)$$

The quasi-maximization problem is then the maximization when the principal faces the menu of quasi-feasible contracts as her constraint set. Similarly, the maximization problem is the principal's maximization when she faces the menu of feasible contracts as a constraint set. The kind of feasibility varies in the research agenda. For instance, in Balder & Yannelis (1993) and Page (1991) the principal solves a maximization problem (see sections 3 and 5), but in Page (1994) the principal solves a quasi-maximization problem (see section 4).

Assume that the rational contract correspondence is defined via some reservation utility function. In Balder (1996) it is not necessary, but it is one of the sufficient condition for the nonemptiness of the feasible set (see proposition 1). Moreover, all the particular cases surveyed here assume this.

2.3 Two existence theorems

Below are the existence theorems as in Balder (1996), which I call quasi-existence theorem and existence theorem, depending on whether they concern with quasi-feasibility or feasibility. The main tool of the proof is the extraction of subsequences in an average sense that characterizes sequential compactness in infinite-dimensional spaces. This tool was introduced by Balder (1989, 1990) and extends to infinite-dimensional spaces the classical Komlós-convergence for a sequence of integrable real-valued functions. Obviously, this does not concerns with this survey in a direct way, but it is worth mentioning that this tool is used in all (with no exceptions) proofs of existence of solutions in this literature.

QUASI-EXISTENCE THEOREM (Balder, 1996): *Assume (A1)-(A4) and suppose that $\Gamma(t) \neq \emptyset$, v-a.s. Then there exists an optimal quasi-feasible contract selection mechanism $f^* \in S(asIR) \cap S(asIC)$ that solves the principal's quasi-maximization problem (asP).*

PROOF: Given (A1)-(A4) and given that $\Gamma(t) \neq \emptyset$, v-a.s., it follows from proposition 1 that $S(asIR) \cap S(asIC) \neq \emptyset$. Then there exists a sequence $\{f_k\} \subset S_{(asIR)} \cap S_{(asIC)}$ such that:

$$\lim_k I_V(f_k) = \sup\{I_V(f) : f \in S_{(asIR)} \cap S_{(asIC)}\}$$

Indeed, this follows from (A4), i.e., given that V is integrably upper bounded on $S(asIR) \cap S(asIC)$ with respect to Γ , and given that $V(t, \cdot)$ is concave and upper semicontinuous on $\Gamma(t)$, and given the measurability of V , the statement above follows immediately from the Fatou lemma in infinite-dimensional spaces.

Given (A1) and (A2), it follows from the Komlós-type Balder theorem (Balder, 1990) that there exists a subsequence $\{f_{k_m}\}$ of $\{f_k\}$, a v-null set $N \in \mathfrak{F}$ (i.e., $v(N) = 0$), and a contract selection mechanism $\varphi^*: T \rightarrow K$ with $\varphi^* \in S(asIR)$ such that $\lim_m s_m(t) = \varphi^*(t)$, $\forall t \in T \setminus N$, where $s_m(t) = \frac{1}{m} \sum_{i=1}^m f_{k_i}(t)$, that is, the subsequence K-converges to φ^* , and moreover, given (A1):

$$I_V(\varphi^*) \geq \lim_k \sup I_V(f_k) = \sup\{I_V(f) : f \in S_{(asIR)} \cap S_{(asIC)}\}$$

Now define the correspondence $L: N \rightarrow P(K)$ by:

$$L(t) = \bigcap_{p=1}^{\infty} \text{cl} \bigcup_{n \geq p} \{s_n(t)\}$$

By lemma 4.1 of Balder (1996), L is $(\mathfrak{F}^{\wedge}, \text{Bor}(P(K)))$ -measurable, where \mathfrak{F}^{\wedge} is the relative σ -field \mathfrak{F} on N . By (A1), K is compact and metrizable for the relative topology. Clearly, $L(t)$ is closed-valued, since it is the countable intersection of closed sets. Since $L(t) \subset K$, it follows that $L(t)$ is itself compact. Moreover, $L(t) \neq \emptyset$. Then L has a measurable selection $\psi^*: N \rightarrow K$

[Castaing & Valadier (1977), theorem III.6]. By the Komlós-type Balder theorem (Balder, 1990), $\psi^* \in \mathcal{S}(asIR)$.

It remains to show that $\psi^* \in \mathcal{S}(asIC)$.

By (A3), $U(t, \cdot)$ is continuous and concave on $\Gamma(t)$. Since $f_{km} \in \mathcal{S}_{(asIR)} \cap \mathcal{S}_{(asIC)}$, $\forall m$, we have that:

$$(*) \quad U(t, s_{km}(t)) \geq U(t, s_{km}(t')), \forall t \in T \setminus N', \forall t' \in M(t), \forall m$$

for some $N' \in \mathcal{T}$ such that $\nu(N')=0$.

Let $t \in T \setminus (N \cup N')$ and $t' \in M(t)$.

If $t' \notin N$, then $s_{km}(t) \rightarrow \varphi^*(t)$ and $s_{km}(t') \rightarrow \varphi^*(t')$, because this happens $\forall t \in T \setminus N$. By assumption (A3), $U(t, \cdot)$ is continuous on $\Gamma(t)$, hence $U(t, s_{km}(t)) \rightarrow U(t, \varphi^*(t))$ and $U(t, s_{km}(t')) \rightarrow U(t, \varphi^*(t'))$. Then, given the inequality above (*), we have that $U(t, \varphi^*(t)) \geq U(t, \varphi^*(t'))$, $\forall t' \in M(t)$, $\forall t \in T \setminus N$, that is, φ^* is IC on $T \setminus N$, so it is asIC, i.e., $\varphi^* \in \mathcal{S}(asIR) \cap \mathcal{S}(asIC)$.

Let $t' \in N$. Then $\psi^*(t')$ is the limit point of $\{s_{km}(t')\}$, by the definition of $L(t')$, so there exists a further subsequence $\{s_{kmj}(t')\}$ such that $s_{kmj}(t') \rightarrow \psi^*(t')$. Moreover, since $t \notin N$, we still have $s_{kmj}(t) \rightarrow \varphi^*(t)$. Then from the continuity of $U(t, \cdot)$ and since the inequality (*) above holds in particular to this further subsequence, i.e.,

$$U(t, s_{kmj}(t)) \geq U(t, s_{kmj}(t')), \forall t \in T \setminus N', \forall t' \in M(t), \forall j,$$

it follows that $U(t, \varphi^*(t)) \geq U(t, \psi^*(t'))$, $\forall t' \in M(t)$, $\forall t \in T \setminus N'$.

Finally, define the contract selection mechanism $f^*: T \rightarrow K$ as:

$$f^*(t) = \varphi^*(t) \chi_{T \setminus N}(t) + \psi^*(t) \chi_N(t)$$

where χ denotes the characteristic function. Then $U(t, f^*(t)) \geq U(t, f^*(t'))$, $\forall t' \in M(t)$, v-a.s., i.e., $f^* \in \mathcal{S}(asIC)$, so $f^* \in \mathcal{S}(asIR) \cap \mathcal{S}(asIC)$, and in addition:

$$I_V(f^*) \geq \lim_k \sup I_V(f_k) = \sup \{I_V(f) : f \in \mathcal{S}_{(asIR)} \cap \mathcal{S}_{(asIC)}\}$$

i.e., f^* solves (asP). ■

EXISTENCE THEOREM (Balder, 1996): *Assume (A1)-(A4) and suppose that $\Gamma(t) \neq \emptyset$, $\forall t \in T$. Then there exists an optimal feasible contract selection mechanism $f^{**} \in \mathcal{S}(IR) \cap \mathcal{S}(IC)$ that solves the principal's maximization problem (P). In particular, f^{**} solves (asP) too.*

PROOF: Given (A1)-(A4) and given that $\Gamma(t) \neq \emptyset, \forall t \in T$, it follows from proposition 1 that $S(IR) \cap S(IC) \neq \emptyset$. Then there exists a sequence $\{f_k\} \subset S_{(IR)} \cap S_{(IC)}$ such that:

$$\lim_k I_V(f_k) = \sup\{I_V(f) : f \in S_{(IR)} \cap S_{(IC)}\}$$

Given (A1) and (A2), it follows from the Komlós-type Balder theorem (Balder, 1990) that there exists a subsequence $\{f_{k_m}\}$ of $\{f_k\}$, a ν -null set $N \in \mathfrak{F}$ (i.e., $\nu(N) = 0$), and a contract selection mechanism $\varphi^*: T \rightarrow K$ with $\varphi^* \in S(asIR)$ such that $\lim_m s_m(t) = \varphi^*(t), \forall t \in T \setminus N$, where $s_m(t) = \frac{1}{m} \sum_{i=1}^m f_{k_i}(t)$, and moreover, the Komlos-type Balder theorem (1990) also says that:

$$I_V(\varphi^*) \geq \lim_k \sup I_V(f_k) = \sup\{I_V(f) : f \in S_{(IR)} \cap S_{(IC)}\}$$

Note that $\{f_{k_m}(t)\} \subset \Gamma(t), \forall t \in T$, so $\varphi^*(t) \in \Gamma(t), \forall t \in T \setminus N$, since $\Gamma(t)$ is closed by (A2).

Now define the correspondence $L: N \rightarrow P(K)$ by:

$$L(t) = \bigcap_{p=1}^{\infty} cl \bigcup_{n \geq p} \{s_n(t)\}$$

By lemma 4.1 of Balder (1996), L is $(\mathfrak{F}^{\wedge}, \text{Bor}(P(K)))$ -measurable, where \mathfrak{F}^{\wedge} is the relative σ -field \mathfrak{F} on N . By (A1), K is compact and metrizable for the relative topology. Clearly, $L(t)$ is closed-valued, since it is the countable intersection of closed sets. Since $L(t) \subset K$, it follows that $L(t)$ is itself compact. Moreover, $L(t) \neq \emptyset$. Then L has a measurable selection $\psi^*: N \rightarrow K$ [Castaing & Valadier (1977), theorem III.6] with $\psi^*(t) \in \text{argmax} \{V(t, x) : x \in L(t)\}$. Define $f^{**}: T \rightarrow K$ by $f^{**}(t) = \varphi^*(t)\chi_{T \setminus N}(t) + \psi^*(t)\chi_N$. Then $f^{**} \in S(IR)$ and $f^{**} = f^*$, ν -a.s. and $I_V(f^{**}) \geq \sup\{I_V(f) : f \in S_{(IR)} \cap S_{(IC)}\}$.

It remains to show that $f^{**} \in S(IC)$.

By (A3) we have that:

$$(*) \quad U(t, s_m(t)) \geq U(t, s_m(t')), \quad \forall t \in T, \forall t' \in M(t), \forall m,$$

provided that $f_{k_i} \in S_{(IC)}, \forall i$.

Let $t \in T$ and $t' \in M(t)$. Then there are four cases to be considered.

Case 1: $t \notin N$ and $t' \notin N$. In this case, $s_m(t) \rightarrow \varphi^*(t)$ and $s_m(t') \rightarrow \varphi^*(t')$. By continuity of $U(t, \cdot)$ and by (*) we have that $U(t, \varphi^*(t)) \geq U(t, \varphi^*(t'))$. Since in this case $\varphi^* = f^{**}$, we get $U(t, f^{**}(t)) \geq U(t, f^{**}(t'))$.

Case 2: $t \notin N$ and $t' \in N$. In this case, $f^{**}(t') = \psi^*(t')$ is still a limit point of $\{s_m(t')\}$, so there exists a subsequence $\{s_{m_j}(t')\}$ converging to $f^{**}(t')$. But at $t \notin N$, $s_{m_j}(t) \rightarrow \varphi^*(t) = f^{**}(t)$. Then by continuity of $U(t, \cdot)$ and by (*) we have that $U(t, f^{**}(t)) \geq U(t, f^{**}(t'))$.

Case 3: $t \in N$ and $t' \notin N$. In this case, $f^{**}(t) = \psi^*(t)$ is still a limit point of $\{s_m(t)\}$, so there exists a subsequence $\{s_{m_j}(t)\}$ converging to $f^{**}(t)$. But at $t' \notin N$, $s_{m_j}(t') \rightarrow \varphi^*(t') = f^{**}(t')$. Then by continuity of $U(t, \cdot)$ and by (*) we have that $U(t, f^{**}(t)) \geq U(t, f^{**}(t'))$.

Case 4: $t \in N$ and $t' \in N$. In this case, $f^{**}(t') = \psi^*(t')$ is still a limit point of $\{s_m(t')\}$, hence working with the same subsequence $\{s_{m_j}(t')\}$ as in case 2 we have $s_{m_j}(t) \rightarrow \psi^*(t) = f^{**}(t)$ and $s_{m_j}(t') \rightarrow \psi^*(t') = f^{**}(t')$, so by continuity of $U(t, \cdot)$ and by (*) we have that $U(t, f^{**}(t)) \geq U(t, f^{**}(t'))$.

Therefore $U(t, f^{**}(t)) \geq U(t, f^{**}(t')), \forall t' \in M(t), \forall t \in T$, i.e., $f^{**} \in S(IC)$.

So $f^{**} \in S(IR) \cap S(IC)$, i.e., f^{**} is feasible and:

$$I_V(f^{**}) \geq \sup\{I_V(f) : f \in S_{(IR)} \cap S_{(IC)}\},$$

hence f^{**} is a solution for the principal's maximization problem (P).

Finally, recall that $f^*(t) = f^{**}(t)$, v-a.s. Then f^{**} is also a solution for (asP), since it is equal to f^* almost surely. ■

All the models surveyed here are mere applications of some of the theorems above. If the model uses quasi-feasibility, then its existence result comes immediately from the quasi-existence theorem (see section 4). If the model uses feasibility, then its existence result comes immediately from the existence theorem (see sections 3 and 5).

3. Contingent contracts with adverse selection

3.1 Introduction

In this section I will show how the model in Balder & Yannelis (1993) is a particular case of the infinite-dimensional principal-agent model. Balder & Yannelis (1993) assume that contracts are contingent to the states of the world. Their model also inherits adverse selection. The principal's problem is to design a contract in such a way that, given the contract mechanism chosen by her, the agent responds by participating and by reporting her type truthfully. Afterwards, the contract pays off randomly.

The notation I used in this section is totally compatible with the infinite-dimensional model described above. The only simplification I made concerns with the probability space that describes the uncertainty about the events of the world. Balder & Yannelis (1993) assume that this probability space can be decomposed into its nonatomic part and its atomic part. All of their existence results are proved separately for the nonatomic and for the atomic part. Here I simply considered an atomless probability space. This means that no single event has positive mass probability. Hence this model fits well for the case of continuously distributed random variables. The reason for doing that is

because I want to consider only the infinite-dimensional and the continuum features of the principal-agent model.

The structure of this section is quite simple. I introduce in section 3.2 the assumptions of Balder & Yannelis (1993)'s model and then I encapsulate them into the assumptions of the general model. In other words, I present the model in this section and show that they imply the assumptions of the general model as in Balder (1996), so that the solution exists. It will be shown that in Balder & Yannelis (1993), the principal solves a maximization problem. So she faces the menu of feasible contracts as constraint set. This model is a particular case in the sense that it assumes an explicit form for the set of contracts K . Specifically, this set is such that contracts pay off randomly.

3.2 The set-up

Let $(\Omega, \mathcal{F}, \mu)$ be a probability space, where \mathcal{F} is σ -field on Ω and μ is a probability measure. The interesting feature of this model is that the probability measure does not need to be atomless, hence the probability space Ω can be decomposed into an atomless part Ω° and an atomic part Ω^1 , i.e., $\Omega = \Omega^\circ \cup \Omega^1$. This introduces the possibility of existence of single events $\{\omega\}$ with strictly positive probability. In the case that μ is absolutely continuous with respect to the Lebesgue measure λ on \mathcal{R} , then its Radon-Nikodym derivative is not continuous. In other words, the beliefs about the events of the world could be represented by discontinuous distribution functions. However, in order to present their model in a simpler way without too much loss of generality, I will assume that the probability space is atomless, that is, I will assume that $\Omega^1 = \emptyset$. This simplification does not rule out the main existing models, since most of them assume that the distributions are continuous with compact support.

The set of potential contract payoffs is described by a separable Banach space $(B, \|\cdot\|)$ where $\|\cdot\|$ is its norm. Assume that its dual is endowed with the weak-star topology. For each state of the world there exists a specific subset of potential contract payoffs available to the players. This is described by the existence of a $(\mathcal{F}, \text{Bor}(P(B)))$ -measurable correspondence $X: \Omega \rightarrow P(B)$, which satisfies the following assumption:

(B1) The correspondence $X: \Omega \rightarrow P(B)$ is integrably bounded, non-empty valued, convex-valued, and weakly compact-valued, μ -a.s.

Then the set of potential contract payoffs is contingent, that is, for each state of the world there might be a different set of potential payoff functions. In section 6 some of these potential payoff functions are provided. The underlying measure is clearly μ . Next we define a state-contingent contract as a measurable selection of the correspondence X . It is assumed that these contracts are Bochner-integrable. Recall that this is an adequate (though not the only one) integral for Banach space valued functions.

DEFINITION: A *state-contingent contract* x is a Bochner-integrable (hence a $(\mathcal{F}, \mathcal{H})$ -measurable) function $x: \Omega \rightarrow B$ such that $x(\omega) \in X(\omega)$, μ -a.s., where \mathcal{H} is the Borel- σ -field on B generated by the weak topology of B . In other words, a state-contingent contract is an Bochner-integrable selection of the correspondence X . Denote by $S^1(X)$ the set of all state-contingent contracts.

To motivate the reader, we could assume for example that the set of potential payoff functions is the set of all measurable nondecreasing functions with values on some bounded set of payoffs. Then for almost every state of the world the principal has to look for a contract that pays off nondecreasingly in some variable, e.g., observable production levels (see section 6 for further examples).

The set of types is described by the probability space (T, \mathcal{I}, ν) , where ν denotes the principal's probability beliefs concerning agent's types. Obviously this probability space is like in Balder (1996)'s model, that is, it atomless. The agent's utility function is a $\mathcal{I} \otimes \mathcal{F} \otimes \mathcal{H}$ -measurable function given by $u: T \times \Omega \times B \rightarrow \mathcal{R}$ and satisfying the following assumption:

(B2) $\forall t \in T$, $u(t, \omega, \cdot): X(\omega) \rightarrow \mathcal{R}$ is continuous on T and affine on $X(\omega)$, μ -a.s. (ii) u is integrably bounded on B with respect to $(T \times \Omega, \mathcal{I} \otimes \mathcal{F}, \nu \otimes \mu)$, where $\mathcal{I} \otimes \mathcal{F}$ is the product- σ -field and $\nu \otimes \mu$ is the product-measure, so that $\exists \xi \in L^1(\nu \otimes \mu)$ such that $\sup\{ |u(t, \omega, x)| : x \in B \} \leq \xi(t, \omega)$, $\nu \otimes \mu$ -a.s.

Assumption (B2-ii) guarantees that the agent's expected utility is finite. Assumption (B2-i) means that the agent is almost surely risk-neutral on the payoffs.

3.3 Fitting the model into the infinite dimensional framework

Given this basic structure of Balder & Yannelis (1993)'s model, I start showing how to fit it into the general framework. Balder (1996) mentions briefly that this model is a particular case of his model, but here I show this deeply.

The set of contracts is given by $K = S^1(X)$. Clearly $K \subset E$, where $E = L^1(\Omega, B)$ is the set of all Bochner-integrable functions from Ω into the Banach space B , and E is endowed with the weak topology τ . Since B is separable, it follows that the weak topology is the coarsest locally convex topology on B compatible with Riesz duality. Then (E, τ) is indeed a Hausdorff locally convex topological space. The economic meaning of the space E is that it is the collection of all (classes of equivalent) state-contingent contracts that have finite mean. Let \mathcal{N} be the Borel- σ -field generated by such a topology. Thus a contract selection mechanism (according to the taxonomy of the infinite-dimensional model) is any $(\mathcal{I}, \mathcal{N})$ -

measurable function $f:T \rightarrow K$. In the following proposition I show that for this particular set of contracts K the assumption (A1) of the general model is satisfied.

PROPOSITION 3: *Let $K = S^1(X)$. Given (B1), K satisfies assumption (A1).*

PROOF: By the assumption (B1), the correspondence X is integrably bounded, and $X(\omega)$ is weakly compact (in the relative topology inherited from E), μ -a.s. By Klei (1988)'s theorem, it is a necessary and sufficient condition for $S^1(X)$ to be weakly compact. Since B is separable, we have that K is metrizable for the (relative) weak topology. It remains to show that K is convex. Let $x, y \in K$, and let $0 \leq \alpha \leq 1$. By assumption, there exists $N \in \wp$ with $\mu(N) = 0$ such that $x(\omega) \in X(\omega)$, $\forall \omega \in \Omega \setminus N$, and there exists $N' \in \wp$ with $\mu(N') = 0$ such that $y(\omega) \in X(\omega)$, $\forall \omega \in \Omega \setminus N'$. Let $N^\circ = N \cup N'$. Then $N^\circ \in \wp$, $\mu(N^\circ) = 0$, and $x(\omega), y(\omega) \in X(\omega)$, $\forall \omega \in \Omega \setminus N^\circ$. Since $X(\omega)$ is convex, μ -a.s., there exists $N^2 \in \wp$ with $\mu(N^2) = 0$ such that $X(\omega)$ is convex, $\forall \omega \in \Omega \setminus N^2$. Now define $N^* = N^\circ \cup N^2$. Then $N^* \in \wp$, $\mu(N^*) = 0$, and $z(\omega) = \alpha x(\omega) + (1-\alpha)y(\omega) \in X(\omega)$, $\forall \omega \in \Omega \setminus N^*$, i.e., μ -a.s. In other words, z is an integrable selection of X , so $z \in K$. Then K is convex. ■

Now we are able to introduce the individual rationality and the rational contract correspondence. The utility of the agent is the $(\mathfrak{F}, \mathfrak{N})$ -measurable function $U: T \times K \rightarrow \mathfrak{R}$ given by:

$$U(t, x) = \int_{\Omega} u(t, \omega, x(\omega)) \mu(d\omega)$$

The rational contract correspondence $\Gamma: T \rightarrow P(K)$ is defined via the reservation utility function $r: T \rightarrow \mathfrak{R}$, which is normalized to be zero, i.e., $r(t) \equiv 0$. Then $\Gamma(t) = \{x \in K: U(t, x) \geq 0\}$. Next I prove that assumptions (A2) and (A3) are satisfied.

PROPOSITION 4: *Assume (B2). Then the rational contract correspondence Γ and the agent's utility $U(t, \cdot)$ satisfy assumptions (A2) and (A3).*

PROOF: Let $t \in T$. I will show that $\Gamma(t)$ is convex and closed and that $U(t, \cdot)$ is affine on $\Gamma(t)$. The proof of continuity will be referred to Balder & Yannelis (1993). Take, $x, y \in \Gamma(t)$ and consider $0 \leq \alpha \leq 1$. Since $x, y \in K$, we have that $x(\omega), y(\omega) \in X(\omega)$, μ -a.s. Moreover $U(t, x) \geq 0$ and $U(t, y) \geq 0$. By proposition 3, we have that $z(\omega) = \alpha x(\omega) + (1-\alpha)y(\omega) \in X(\omega)$, μ -a.s. By assumption (B2), $u(t, \omega, \cdot)$ is affine on $X(\omega)$, μ -a.s. Then it can be written as $u(t, \omega, z(\omega)) = a(t, \omega) + b(t, \omega)z(\omega)$, where the functions a and b are integrably bounded. Then:

$$\begin{aligned}
U(t, z) &= \int_{\Omega} u(t, \omega, z(\omega)) \mu(d\omega) \\
&= \int_{\Omega} [a(t, \omega) + b(t, \omega)z(\omega)] \mu(d\omega) \\
&= \int_{\Omega} a(t, \omega) \mu(d\omega) + \int_{\Omega} b(t, \omega)(\alpha x(\omega) + (1-\alpha)y(\omega)) \mu(d\omega) \\
&= \alpha \int_{\Omega} [a(t, \omega) + b(t, \omega)x(\omega)] \mu(d\omega) + (1-\alpha) \int_{\Omega} [a(t, \omega) + b(t, \omega)y(\omega)] \mu(d\omega) \\
&= \alpha U(t, x) + (1-\alpha)U(t, y) \geq 0
\end{aligned}$$

Then $z \in \Gamma(t)$, so $\Gamma(t)$ is convex, and also $U(t, \cdot)$ is affine on $\Gamma(t)$.

The continuity of $U(t, \cdot)$ on K follows from the bounded integrability assumption [Balder & Yannelis (1993, prop.4.3 and prop.4.7)]. The function U is finite-valued because of the bounded integrability.

Now the closedness of $\Gamma(t)$ follows trivially from the continuity of $U(t, \cdot)$ and from the very definition of $\Gamma(t)$. ■

In Balder & Yannelis (1993), the agent faces no *a priori* restrictions on the possibility for misreporting, hence the misspecification correspondence $M: T \rightarrow P(T)$ is trivially set as $M(t) = T$. The only restrictions come *a posteriori* from the incentive compatibility conditions for the principal's maximization problem. Furthermore, the individual rationality and the incentive compatibility constraints hold everywhere. In other words, there does not exist a set of types with probability zero on which the agent is free to not obey the incentive compatibility nor the individual rationality constraints. Therefore, Balder & Yannelis (1993) work with menus of feasible contracts.

The principal's utility random function is given by a $\mathfrak{F} \otimes \mathfrak{G} \otimes \mathfrak{H}$ -measurable function $v: T \times \Omega \times \mathfrak{X} \rightarrow \mathfrak{R}$ satisfying the following assumption:

(B3) (i) $\forall t \in T, v(t, \omega, \cdot): X(\omega) \rightarrow \mathfrak{R}$ is concave and upper semicontinuous on $X(\omega)$, μ -a.s. (ii) v is upper integrably bounded on \mathfrak{X} with respect to $(T \times \Omega, \mathfrak{F} \otimes \mathfrak{G}, \nu \otimes \mu)$, where $\mathfrak{F} \otimes \mathfrak{G}$ is the product- σ -field and $\nu \otimes \mu$ is the product-measure, so that $\exists \psi \in L^1(\nu \otimes \mu)$ such that $\sup\{v(t, \omega, x): x \in B\} \leq \psi(t, \omega)$, $\nu \otimes \mu$ -a.s.

Assumption (B3-i) means that the principal may be almost surely risk averse on the payoffs. Obviously this is in a weak sense, since she can be risk neutral too, should her utility be linear. Assumption (B3-ii) guarantees that her expected utility over types is finite.

Given a type t and a contract $x \in S^1(X)$, the expected utility of the principal is the function $V: T \times K \rightarrow [-\infty, \infty)$ given by:

$$V(t, x) = \int_{\Omega} v(t, \omega, x(\omega)) \mu(d\omega)$$

Notice that the model begins with utilities that grow up to the utilities as in Balder (1996) through integration, that is, through taking expectations. This is a common aspect of many particular cases. They refine the economic environment and through integration this refinement becomes coarser enough to meet the utilities considered in the general model of Balder (1996). Next I show that assumption (A4) is satisfied.

PROPOSITION 5: *Assume (B3). Then the principal's utility V satisfies assumption (A4).*

PROOF: Let $f \in \Gamma(t) \mathcal{S}(\mathbb{R}) \cap \mathcal{S}(\mathbb{C})$. By assumption (B3), v is upper integrably upper bounded on B with respect to $(T \times \Omega, \mathfrak{F} \otimes \mathcal{F}, \nu \otimes \mu)$, so $\exists \psi \in L^1(\nu \otimes \mu)$ such that $\sup\{v(t, \omega, x) : x \in B\} \leq \psi(t, \omega)$, $\nu \otimes \mu$ -a.s. Therefore:

$$V(t, f(t)) = \int_{\Omega} v(t, \omega, f(\omega)) \mu(d\omega) \leq \int_{\Omega} \sup\{v(t, \omega, x) : x \in B\} \mu(d\omega) \leq \int_{\Omega} \psi(t, \omega) \mu(d\omega)$$

Consider $\zeta(t) = \int_{\Omega} \psi(t, \omega) \mu(d\omega)$. Since ψ is jointly-measurable and integrable, it follows from Fubini's theorem that:

$$\int_T \zeta(t) \nu(dt) = \int_T \int_{\Omega} \psi(t, \omega) \mu(d\omega) \nu(dt) = \int_{T \times \Omega} \psi(t, \omega) (\mu \otimes \nu)(d\omega, dt) < \infty$$

Therefore $\int_T V(t, f(t)) \nu(dt) \leq \int_T \zeta(t) \nu(dt) < \infty$. Hence V is integrably upper bounded on $\mathcal{S}(\mathbb{R}) \cap \mathcal{S}(\mathbb{C})$ with respect to T . This proves (A4-iii).

Let $x, y \in \Gamma(t)$ and take $z(\omega) = \alpha x(\omega) + (1 - \alpha)y(\omega)$, where $0 \leq \alpha \leq 1$. By concavity we have:

$$\begin{aligned} V(t, z) &= \int_{\Omega} v(t, \omega, \alpha x(\omega) + (1 - \alpha)y(\omega)) \mu(d\omega) \\ &\geq \alpha \int_{\Omega} v(t, \omega, x(\omega)) \mu(d\omega) + (1 - \alpha) \int_{\Omega} v(t, \omega, y(\omega)) \mu(d\omega) \\ &= \alpha V(t, x) + (1 - \alpha)V(t, y) \end{aligned}$$

So $V(t, \cdot)$ is concave on $\Gamma(t)$. Since the function $v(t, \omega, \cdot) : X(\omega) \rightarrow \mathcal{R}$ is upper semicontinuous on $X(\omega)$, μ -a.s., $\forall t \in T$, we have that $\exists N \in \mathcal{F}$ with $\mu(N) = 0$ such that $\{x \in X(\omega) : v(t, \omega, x) \geq v(t, \omega, x(\omega))\}$ is (relatively) closed, $\forall \omega \in \Omega \setminus N$. Since $(\Omega, \mathcal{F}, \mu)$ is atomless, the integral over Ω is equal to the integral over $\Omega \setminus N$. Since B is endowed with the weak topology and its topological dual B^* is endowed with the weak-star topology, the pair (B, B^*) forms a (Riesz) dual system, hence the evaluation map defined by the integral is jointly continuous, so V is also upper semicontinuous. This proves (A4-ii). The \mathfrak{F} -measurability of $V(t, f(t))$ follows easily. ■

Mutatis mutandis, Balder & Yannelis (1993) pose therefore the principal's problem as:

$$\begin{aligned} \sup_{f \text{ is measurable}} I_V(f) &= \int_T V(t, f(t)) \nu(dt) \\ \text{s.t. } U(t, f(t)) &\geq U(t, f(t')), \quad \forall t, t' \in T && (\text{incentive compatibility}) \\ U(t, f(t)) &\geq 0, \quad \forall t \in T && (\text{individual rationality}) \end{aligned}$$

In other words, in the Balder & Yannelis (1993)'s model, the principal faces the menu of feasible contracts as her constraint set and hence solves the maximization problem:

$$(P) \quad \sup_{f \in \mathcal{S}_{(IR)} \cap \mathcal{S}_{(IC)}} I_V(f)$$

Since all the assumptions of the infinite-dimensional principal-agent model are satisfied, it follows from the existence theorem that a solution exists, i.e., there exists an optimal state-contingent contract selection mechanism that solves the principal's maximization problem.

4. Mixed contracts with many agents

4.1 Introduction

This section presents the model of mixed contracts in the principal multi-agent model as in Page (1994). This is a model where the principal has incomplete information about the types of many agents and where these agents compete for a contract and the principal selects an agent via a contract auction. In this multi-agent setting, the equilibrium notion is that of dominant strategy almost surely incentive compatible and almost surely individually rational mechanism.

Again I state the assumptions of the model and show that they satisfy the assumptions of the general model as in Balder (1996), so that the solution exists. Actually, as will be inferred from the model, I show that in Page (1994) the principal solves a quasi-maximization problem.

Page (1994) assumes that the principal and the agents might be risk averse. As pointed out in last section, this model also begins with specific economic environment and grows up to the general model through integration. The risk aversion appears in the first steps, precisely on the monetary payoffs, but at the end it is shown that the expected utility over types is linear on the rational contract correspondence. This is because of the very nature of mixed contracts, that is, because they are probability measures, and of course, integration is linear on the measures. Then risk aversion (or risk neutrality) here is a characteristic of the principal and the agents that appears on monetary payoffs and this aspect disappears after taking expectations.

He also assumes that the set of (quasi-) feasible contracts is uniformly bounded, hence whatever the contract awarded by the principal, both the principal and the winning agent have limited liability. This excludes the possibility of linear contracts, as usual in the standard literature [Page (1994), p. 25-26]. Due to incomplete information, the principal faces an adverse selection problem. In Page (1994), her probability beliefs concerning agents' type profiles allow for correlation between agents' types. For each type profile, the corresponding contract mechanism is a probability measure on the set of agent-contract pairs. Therefore, as told before, this is a model of mixed contracts.

4.2 The set-up

There is a finite set $I = \{1, \dots, m\}$ of agents. The set of agents is endowed with the discrete topology generated by the metric $\rho(i, i') = 1$, if $i = i'$, and zero otherwise. Therefore the Borel- σ -field generated by this metrizable topology is the trivial σ -field $\mathbb{T} = \{\emptyset, I\}$. Moreover (I, ρ) is a compact metric space. The most economic significant feature of the set of agents is the triviality of the σ -field \mathbb{T} . It means that no coalition is allowed, unless it is the whole set of agents, so it captures the idea that agents compete for a contract in the contract auction. It might be argued that this is not the best model for competition among agents. For instance, according to the general equilibrium literature, competition is better modeled by taking an infinite number of agents, namely, a continuum. Then, even though the agents in the latter case are allowed to form coalitions, the final outcome of this possibly cooperative system coincides with the competitive outcome. Nevertheless, should this be the case for the model presented in this section, there would be no significance at all in awarding the contract to one winning single agent, since this agent would have measure zero. So, reallocating the contract to any other single agent would cause no effect to the final outcome. Therefore, finiteness of the set of agents is required because otherwise this special principal-multi-agent model would be senseless. I conjecture that the only possibility to introduce a continuum of agents in the principal multi-agent contract auction is by redefining the contract auction in such a way that the principal awards the contract to a winning coalition of positive measure.

The set of types for each agent is a measurable space (T_i, \mathfrak{T}_i) . The principal faces the set of type profiles according to the atomless probability space (T, \mathfrak{T}, ν) , where $T = \times_{i \in I} T_i$ is the set of type profiles, $\mathfrak{T} = \otimes_{i \in I} \mathfrak{T}_i$ is the product σ -field on T , and ν is a probability measure on \mathbb{T} representing the principal's probability beliefs concerning agents' types. As usual, for the cases where agent i and the remaining agents have to be seen separately, we write $t = (t_i, t_{-i}) = (t_1, \dots, t_{i-1}, t_i, t_{i+1}, \dots, t_m)$ for the types profile, $T_{-i} = \times_{j \neq i} T_j$ for the type space, and $\mathfrak{T}_{-i} = \otimes_{j \neq i} \mathfrak{T}_j$ for the σ -field of type profiles. This type space captures the idea of adverse selection faced by the principal in the contract auction.

The set of actions for agent i is described by a compact metric space (A_i, δ_i) . Let \mathbb{D}_i be the Borel- σ -field generated by the (metrizable) topology generated by the metric δ_i

on the action space. Hence the set of actions can be a continuum, for example a compact interval. Moral hazard appears since the individual actions affect principal's utility somehow.

The utility of agent i is a $\beta \otimes \mathbf{D}_i \otimes \mathfrak{T}_i$ -measurable function $u_i : \mathfrak{R} \times A_i \times T_i \rightarrow \mathfrak{R}$ (where β is the Borel- σ -field of the usual topology on \mathfrak{R}) defined over monetary payoffs, actions, and types, and satisfying the following assumption:

(C1) (i) $\forall (a_i, t_i) \in A_i \times T_i, u_i(\cdot, a_i, t_i) : \mathfrak{R} \rightarrow \mathfrak{R}$ is a concave and increasing function; (ii) $\forall t_i \in T_i, u_i(\cdot, \cdot, t_i) : \mathfrak{R} \times A_i \rightarrow \mathfrak{R}$ is continuous.

Assumption (C1-i) means that the agent likes money and might be risk averse over the monetary payoffs. Of course this does not exclude risk neutrality, since her utility is not necessarily strict concave. The remaining part is a technical assumption. In order to describe the reservation utility of each agent, Page (1994) introduces a reservation utility function given by a bounded and $\mathbf{I} \otimes \mathbf{T}$ -measurable function $r : I \times T \rightarrow \mathfrak{R}$. Since I is endowed with the discrete topology, the reservation utility function is trivially continuous on I .

The principal's utility function is described by the β -measurable function $v : \mathfrak{R} \rightarrow \mathfrak{R}$ defined over the monetary payoffs and satisfying the following assumption:

(C2) v is concave, increasing, and differentiable.

Again the interpretation of this assumption is straightforward. It means that the principal likes money and is risk averse. The differentiability assumption is technical.

Let $Meas(\mathfrak{R}, \mathfrak{R})$ be the set of all real-valued β -measurable functions. The set of feasible contracts is given by a subset $\Phi \subset Meas(\mathfrak{R}, \mathfrak{R})$ satisfying the following assumption:

(C3) (i) $\exists [\underline{\theta}, \bar{\theta}] \subset \mathfrak{R}$ such that, $\forall s \in \Phi$ and $\forall x \in \mathfrak{R}, s(x) \in [\underline{\theta}, \bar{\theta}]$. (ii) Φ is weakly-sequentially compact, i.e., $\forall \{s_n\} \subset \Phi, \exists \{s_{n_k}\} \subset \{s_n\}$ and $\exists s \in \Phi$ such that $s_{n_k}(x) \rightarrow s(x), \lambda - a.e.$, where λ is the Lebesgue measure on \mathfrak{R} . (iii) Φ is separated on \mathfrak{R} , i.e., if $s_1, s_2 \in \Phi$ are such that $s_1(x) \neq s_2(x)$ for some $x \in \mathfrak{R}$, then $\lambda(\{x \in \mathfrak{R} / s_1(x) \neq s_2(x)\}) > 0$.

Assumption (C3-i) means that the payoffs of all contracts are uniformly bounded, that is, whatever the contract and whatever the monetary outcome, no one expects payoffs lower than $\underline{\theta}$ or higher than $\bar{\theta}$. Assumption (C3-iii) means that there does not exist

redundant contracts. Two contracts are *redundant* if they yield the same payoffs almost everywhere.

Consider the following metric on Φ :

$$d(s_1, s_2) = \int_{(-\infty, \infty)} |s_1(x) - s_2(x)| \lambda(dx)$$

Then (Φ, d) is a compact metric space. In the definition of the metric d above, Page (1994) considers a probability measure other than the Lebesgue measure λ . However, he assumes that the former measure is equivalent to the latter one, the Lebesgue measure, in the sense that they are absolutely continuous with respect to each other, so that they have the same set of null events. Here I considered the metric d defined via the Lebesgue measure, because I think there is no loss of generality in doing that. Indeed, they generate the same metric d , hence the same topology. Let τ_d be the topology generated by the metric d , and let \mathfrak{D} be the Borel- σ -field generated by the topology. Consider $I \times \Phi$ endowed with the product- σ -field $\mathfrak{T} \otimes \mathfrak{D}$. Let $Prob(I \times \Phi)$ be the set of all probability measures on $I \times \Phi$. Since I and Φ are compact metric spaces, it follows from Tychonoff's theorem that $I \times \Phi$ is compact, hence $Prob(I \times \Phi)$ is compact and metrizable for the weak topology [Parthasarathy (1967)]. Let Π be the Borel- σ -field generated by the weak topology on $Prob(I \times \Phi)$.

DEFINITION: A *direct auction mechanism* is any $\mathfrak{T} \otimes \Pi$ -measurable function $f: T \rightarrow Prob(I \times \Phi)$. Let $Meas(T, Prob(I \times \Phi))$ be the set of all direct auction mechanisms, i.e., the set of all measurable functions from T to $Prob(I \times \Phi)$.

Then a direct auction mechanism assigns to each type profile a probability distribution on the contract space for each agent. Under this probability distribution, the principal allocates the contract to the winning agent. In this sense, the contract is said to be a mixed contract by the very reason that it is a probability distribution. In the case the probability distribution is some Dirac measure, that is, concentrated on some singleton point, it is called a *pure contract* [Page (1992)]. According to the notation introduced in the general model, set $K = Meas(T, Prob(I \times \Phi))$ or, equivalently (up to a measurable transformation), $K = Prob(I \times \Phi)$. In the subsection 4.3 below I show that this model is a particular case of Balder (1996)'s. There are two assumptions left, assumptions (C4) and (C5), whose statements have been postponed to subsection 4.3. This is because (C4) concerns specifically to the behavior of the individual density functions on the payoffs and its role is just to smooth the integration (expectation) procedure necessary to turn utilities into the framework of Balder (1996)'s model. Assumption (C5) is concerned in turn to nontrivial solutions.

4.3 Fitting the model into the infinite dimensional framework

I start this subsection by showing that the contract space given by Page (1994) satisfies the assumption (A1). Then I do the same for all the remaining assumptions step by step.

PROPOSITION 6: *Assume (C3). Then $K = \text{Meas}(T, \text{Prob}(I \times \Phi))$ satisfies (A1).*

PROOF: I will show that K is Komlós-sequentially compact. Let $\{f_n\} \subset M(T, \text{Prob}(I \times \Phi))$ be a sequence of direct auction mechanisms. Then $f_n(t) \in \text{Prob}(I \times \Phi)$, $\forall n$, i.e., $\{f_n(t)\}$ is a sequence of probability measures on $I \times \Phi$. As I already argued, (C3) implies that $\text{Prob}(I \times \Phi)$ is weakly compact, hence there exists a subsequence $\{f_{n_k}(t)\}$ of $\{f_n(t)\}$ and there exists a probability measure $f(t)$ such that $f_{n_k}(t) \rightarrow_w f(t)$, i.e., $f_{n_k}(t)$ converges weakly to $f(t)$. Define the sequence of averaged direct auction mechanisms $f^k(t) = \frac{1}{k} \sum_{j=1}^k f_{n_j}(t)$. Clearly $f^k(t) \in \text{Prob}(I \times \Phi)$, $\forall k$, since $\text{Prob}(I \times \Phi)$ is convex. Recall that the space $C[I \times \Phi]$ of all real-valued continuous functions on $I \times \Phi$ is the topological dual of $\text{Prob}(I \times \Phi)$. Then by the very definition of weak convergence, we have that $\forall h \in C[I \times \Phi]$:

$$\lim_{k \uparrow \infty} \int_{I \times \Phi} h(i, s) f_{n_k}(t)(d(i, s)) = \int_{I \times \Phi} h(i, s) f(t)(d(i, s)), \quad v - a.s.$$

Then $\forall \varepsilon > 0$, $\exists N > 0$ such that $\forall n_j \geq N$:

$$\left\| \int_{I \times \Phi} h(i, s) f_{n_j}(t)(d(i, s)) - \int_{I \times \Phi} h(i, s) f(t)(d(i, s)) \right\| \leq \varepsilon$$

Then for the sequence of averaged direct auction mechanisms we have that for k large enough:

$$\begin{aligned} & \left\| \frac{1}{k} \sum_{j=1}^k \int_{I \times \Phi} h(i, s) f_{n_j}(t)(d(i, s)) - \int_{I \times \Phi} h(i, s) f(t)(d(i, s)) \right\| = \\ & = \left\| \frac{1}{k} \sum_{j=1}^k \left[\int_{I \times \Phi} h(i, s) f_{n_j}(t)(d(i, s)) - \int_{I \times \Phi} h(i, s) f(t)(d(i, s)) \right] \right\| \\ & \leq \frac{1}{k} \sum_{j=1}^k \left\| \int_{I \times \Phi} h(i, s) f_{n_j}(t)(d(i, s)) - \int_{I \times \Phi} h(i, s) f(t)(d(i, s)) \right\| \\ & \leq \frac{1}{k} \sum_{j=1}^k \varepsilon \\ & = \varepsilon \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have that:

$$\lim_{k \uparrow \infty} \int_{I \times \Phi} h(i, s) f^k(t)(d(i, s)) = \int_{I \times \Phi} h(i, s) f(t)(d(i, s)), \quad v - a.s.$$

In other words, the sequence of averaged subsequence of direct auction mechanisms converges weakly to $f(t)$, hence the sequence of direct auction mechanisms Komlós-converges to $f(t)$, v-a.s. This proves that $Prob(I \times \Phi)$ is Komlós-sequentially weakly compact.

Now define the map $b: T \times I \times \Phi \rightarrow [0, \infty]$ by $b(t, i, s) = 1$. Then b is clearly $\mathfrak{I} \otimes \mathfrak{T} \otimes \mathfrak{D}$ -measurable. Moreover, $\forall t \in T$ and $\forall \gamma \in \mathfrak{R}$, the set $\mathcal{K} = \{(i, s) \in I \times \Phi / b(t, i, s) \leq \gamma\}$ is weakly compact. Indeed, since b is constant, it is continuous, hence \mathcal{K} is weakly closed. Since $I \times \Phi$ is weakly compact, it follows that \mathcal{K} is itself weakly compact. This then means that b is inf-compact. Finally:

$$\sup \left\{ \int_T \int_{I \times \Phi} h(t, i, s) \xi(d(i, s) / t) \nu(dt) : \xi \in \text{Meas}(T, Prob(I \times \Phi)) \right\} = 1 < \infty$$

Therefore, $K = \mathcal{M}(T, Prob(I \times \Phi))$ is v-tight. Then by the generalized Prohorov's theorem [Balder (1990), theorem 5.1, or Balder (1991), theorem A.5], it follows that $\text{Meas}(T, Prob(I \times \Phi))$ is Komlós weakly compact. In this case, the Hausdorff locally convex topological vector space E the set K belongs to is K itself, so K is trivially compact and metrizable for the relative topology inherited from E .

Finally, let $f, g \in K$ and take $0 \leq \alpha \leq 1$. Then the map $h(t) = \alpha f(t) + (1 - \alpha)g(t)$ is well defined, since $Prob(I \times \Phi)$ is convex. It is obviously measurable, so K is convex. ■

Each agent i has a probability density function describing the random nature of the monetary outcome. This belief is assumed to be private in the sense that it depends on her individual action space and her individual type set only. Specifically, for each $(a_i, t_i) \in A_i \times T_i$, agent i computes her expected utility according to a probability measure ζ_i with Radon-Nikodym derivative given by the density function $\eta_i(\cdot | a_i, t_i)$ defined on \mathfrak{R} and satisfying, $\forall i = 1, \dots, m$, the following assumption:

(C4) (i) $\exists x^* > -\infty$ such that $\text{supp} \eta_i(\cdot | a_i, t_i) \subset [x^*, \infty)$, $\forall (a_i, t_i) \in A_i \times T_i$. (ii) $\forall (a_i, t_i) \in A_i \times T_i$, $\text{id}: \mathfrak{R} \rightarrow \mathfrak{R} \in L^2(\zeta_i)$; (iii) $\forall (x, t_i) \in \mathfrak{R} \times T_i$, $\eta_i(x | \cdot, t_i)$ is continuous on A_i , and $\forall a_i \in A_i$, $\eta_i(\cdot | a_i, \cdot)$ is $\beta \otimes \mathfrak{I}_i$ -measurable on $\mathfrak{R} \times T_i$.

Assumption (C4-i) means that the supports of the density functions are uniformly bounded from below by some value x^* . Despite it excludes, e.g., the case where the beliefs of the agents with respect to the monetary payoffs are normally distributed, it is not so restrictive at all, because it is reasonable to assume that no monetary can be infinitely negative. Assumption (C4-ii) means that the identity function on \mathfrak{R} has finite mean and variance under the measure $\eta_i(\cdot | a_i, t_i)$. Assumption (C4-iii) is technical.

Page (1994) interprets the density function $\eta_i(\cdot|a_i, t_i)$ as a random production technology available to agent i , given that the agent i is type t_i and takes action a_i .

Suppose that the principal implements a contract $s \in \Phi$ to the agent i . If agent i is type $t_i \in T_i$ and takes the action $a_i \in A_i$, then her expected utility under the contract s is given by:

$$E_i u_i(s|a_i, t_i) = \int_{(-\infty, \infty)} u_i(s(x), a_i, t_i) \zeta_i(dx) = \int_{[x^*, \infty)} u_i(s(x), a_i, t_i) \eta_i(x|a_i, t_i) \lambda(dx)$$

where $E_i \equiv E_{\zeta_i}$ denotes the expectation taken with respect to the probability measure ζ_i . If such an agent wins the auction and is awarded contract $s \in \Phi$, then the agent chooses an optimal action $a_i^*(s, t_i) \in \arg \sup_{a_i \in A_i} E_i u_i(s|a_i, t_i)$. Let $u_i^*(s, t_i) = u_i(s|a_i^*(s, t_i), t_i)$ be the type t_i agent i 's optimal expected utility under contract $s \in \Phi$. The *best response correspondence* $B_i : \Phi \times T_i \rightarrow P(A_i)$ of agent i to a contract $s \in \Phi$, given her type, is defined by:

$$B_i(s, t_i) = \{a_i \in A_i / E_i u_i(s|a_i, t_i) \geq u_i^*(s, t_i)\}$$

Let $u^*(i, s, t_i) = \sup\{E_i u_i(s|a_i, t_i) : a_i \in A_i\}$ be the optimal expected utility of agent i . Then the collection of agents' optimal expected utilities can be viewed as a functional $u^* : I \times \Phi \times T_i \rightarrow [-\infty, \infty]$.

Under the assumptions stated in the model, we conclude that the best response correspondence of the winning agent is well behaved, that is, it is nonempty, compact-valued, and upper semicontinuous. This is proved in proposition 7 below.

PROPOSITION 7: *Assume that (C1) and (C4) hold. Then the winning agent i 's best response correspondence $B_i : \Phi \times T_i \rightarrow P(A_i)$ is $(\mathbb{D} \otimes \mathfrak{S}_i, \mathbb{D}_i)$ -measurable and is nonempty and compact-valued. Moreover, $\forall t_i \in T_i, B_i(\cdot, t_i)$ is upper semicontinuous on Φ .*

PROOF: By assumption (C1), $\forall t_i \in T_i, u_i(\cdot, \cdot, t_i) : \mathfrak{R} \times A_i \rightarrow \mathfrak{R}$ is continuous. In particular, $\forall a_i \in A_i, u_i(\cdot, a_i, t_i) : \mathfrak{R} \rightarrow \mathfrak{R}$ is continuous. By assumption (C4), $\forall (a_i, t_i) \in A_i \times T_i, u_i(\cdot, a_i, t_i) \in L^2(\zeta_i)$, then in particular, $u_i(\cdot, a_i, t_i)$ is integrable, so the function $a_i \mapsto E_i u_i(s|a_i, t_i)$ is well defined and is continuous on $A_i, \forall t_i \in T_i$. Since A_i is compact, it follows from Weierstrass theorem that $\arg \sup_{a_i \in A_i} E_i u_i(s|a_i, t_i) \neq \emptyset$. Then $B_i(s, t_i) \neq \emptyset, \forall (s, t_i) \in \Phi \times T_i$.

By assumption (C4), $\forall a_i \in A_i, \eta_i(\cdot|a_i, \cdot)$ is $\beta \otimes \mathfrak{S}_i$ -measurable on $\mathfrak{R} \times T_i$. Since $u_i : \mathfrak{R} \times A_i \times T_i \rightarrow \mathfrak{R}$ is a $\beta \otimes \mathbb{D}_i \otimes \mathfrak{S}_i$ -measurable function, it follows that, $\forall a_i \in A_i,$

$E_i u_i(s|a_i, t_i)$ is $(\mathbb{F} \otimes \mathfrak{S}_i, \mathbf{D}_i)$ -measurable. It is also integrable, since $\forall (a_i, t_i) \in A_i \times T_i$, $u_i(\cdot, a_i, t_i) \in L^2(\zeta_i)$. Furthermore, $\forall (s, t_i) \in \Phi \times T_i$, $E_i u_i(s|a_i, t_i)$ is continuous. Since A_i is a compact metric space, it follows that A_i is complete and totally bounded [Aliprantis & Border (1994), theorem 3.20]. Then we can find a countable collection of finite subsets A_i^n of A_i such that $A_i = \cup_{a_i \in A_i^n} B_{\frac{1}{n}}(a_i)$, where $B_{\frac{1}{n}}(x)$ denotes the open ball around x with radius $1/n$. Now notice that $F = \cup_n A_i^n$ is countable (since it is the countable union of countable sets) and is dense in A_i . Hence A_i is separable. Therefore, given that A_i is separable, it follows that $B_i : \Phi \times T_i \rightarrow P(A_i)$ is $(\mathbb{D} \otimes \mathfrak{S}_i, \mathbf{D}_i)$ -measurable [Castaing & Valadier (1977), lemma III-14].

Given that $a_i \mapsto E_i u_i(s|a_i, t_i)$ is continuous, $B_i(s, t_i) = \{a_i \in A_i / E_i u_i(s|a_i, t_i) \geq u_i^*(s, t_i)\}$ is closed. Since A_i is compact, we have that $B_i(s, t_i) \subset A_i$ is compact.

It remains to show that, $\forall t_i \in T_i$, $B_i(\cdot, t_i)$ is upper semicontinuous on Φ . But this follows immediately from Berge's maximum theorem [Aliprantis & Border (1994), theorem 14.30]. This finishes the proof. ■

Suppose that agent i wins the auction and the contract s is awarded. Then the monetary outcome will be generated by the random production technology $\eta_i(\cdot|a_i, t_i)$ if the agent i is type t_i and take the action a_i . Thus the principal's expected payoff functional $E_i v(\cdot, \cdot) : \Phi \times A_i \times T_i \rightarrow [-\infty, \infty]$ is given by:

$$\begin{aligned} E_i v(id - s, a_i, t_i) &= \int_{(-\infty, \infty)} v(x - s(x)) \zeta_i(dx | a_i, t_i) \\ &= \int_{[x^*, \infty)} v(x - s(x)) \eta_i(x | a_i, t_i) \lambda(dx) \end{aligned}$$

where $id - s$ is the function given by $(id - s)(x) = x - s(x)$. Therefore, once the agent i wins the auction, and given that the contract rule is s , the principal observes the output x generated by agent i 's random production function $\eta_i(\cdot|a_i, t_i)$, pays $s(x)$ to the agent, and finally gets the profit $x - s(x)$ that gives the principal the expected utility defined above.

The importance of following proposition is that it is the first step to prove later on that the principal's utility behaves like the utility of the general model of Balder (1996).

PROPOSITION 8 (Page, 1994): *Assume that (C2)-(C4) hold. Then the principal's expected utility functional $E_i v(\cdot, \cdot) : \Phi \times A_i \times T_i \rightarrow [-\infty, \infty]$ is bounded on $\Phi \times A_i \times T_i$, continuous on $\Phi \times A_i$, $\forall t_i \in T_i$, and $\mathbb{D} \otimes \mathbf{D}_i \otimes \mathfrak{S}_i$ -measurable.*

PROOF: Given (C2), we have that, $\forall (s, a_i, t_i) \in \Phi \times A_i \times T_i$ and $\forall E \in \beta$:

$$\begin{aligned} \int_E v(x^* - H) \eta_i(x | a_i, t_i) \lambda(dx) &\leq \int_E v(x^* - H) \eta_i(x | a_i, t_i) \lambda(dx) \\ &\leq \int_E [v(0) + v'(0)(x - s(x))] \eta_i(x | a_i, t_i) \lambda(dx) \end{aligned}$$

Indeed, the first inequality holds because v is increasing; the second inequality holds because v is differentiable and concave. By assumption (C4), the (absolute) means of the density $\eta_i(x | a_i, t_i)$ are uniformly bounded. This implies that if we take any $L \geq \int_{[x^*, \infty)} |x| \eta_i(x | a_i, t_i) \lambda(dx)$, then we get $\int_E |x| \eta_i(x | a_i, t_i) \lambda(dx) \leq L$, $\forall E \in \beta$. By assumption (C3), the set Φ of contract payoffs is uniformly bounded. It means that $\exists Q > 0$ such that $\|x - s(x)\| \leq Q$, $\forall E \in \beta$. Indeed, by assumption (C3-i), we can take any $Q \geq \max\{\inf E, \sup E\} + 2(\bar{\theta} - \underline{\theta})$. Then we have that $E_i v(\cdot, \cdot, \cdot) : \Phi \times A_i \times T_i \rightarrow [-\infty, \infty]$ is bounded on $\Phi \times A_i \times T_i$. Indeed:

$$\begin{aligned} \|\otimes_i v(s, a_i, t_i)\| &\leq \int_E \|v(x^* - H)\| \eta_i(x | a_i, t_i) \lambda(dx) \\ &\leq \int_E \|v(0) + v'(0)(x - s(x))\| \eta_i(x | a_i, t_i) \lambda(dx) \\ &\leq \|v(0)\| + \|v'(0)\| (L + Q) < \infty \end{aligned}$$

This proves that $E_i v(\cdot, \cdot, \cdot) : \Phi \times A_i \times T_i \rightarrow [-\infty, \infty]$ is bounded on $\Phi \times A_i \times T_i$.

Now fix $t_i \in T_i$ and take a sequence $\{(s^n, a_i^n)\} \subset \Phi \times A_i$ converging to (s, a_i) in the product metric $d \times \delta_i$. It follows from (C2), (C3), and (C4) (using the same argument as above) that $\forall c > 0$ and $\forall n$:

$$\begin{aligned} v^*(x - H) \int_{[|x| > c]} \eta_i(x | a_i^n, t_i) \lambda(dx) &\leq \int_{[|x| > c]} v(x^* - s^n(x)) \eta_i(x | a_i^n, t_i) \lambda(dx) \\ &\leq \int_{[|x| > c]} [v(0) + v'(0)(x - s^n(x))] \eta_i(x | a_i^n, t_i) \lambda(dx) \\ &\leq (v(0) - v'(0)\underline{\theta}) \int_{[|x| > c]} \eta_i(x | a_i^n, t_i) \lambda(dx) + v'(0) \int_{[|x| > c]} |x| \eta_i(x | a_i^n, t_i) \lambda(dx) \\ &\leq (v(0) - v'(0)\underline{\theta}) \int_{[|x| > c]} \eta_i(x | a_i^n, t_i) \lambda(dx) + v'(0) \frac{1}{c^2} \int_{[|x| > c]} |x^2| \eta_i(x | a_i^n, t_i) \lambda(dx) \end{aligned}$$

The second inequality comes from differentiability and from concavity of v . The last inequality comes from the Tchebyshev inequality applied to the second integral. By assumption (C4-ii), the variances according to the measures $\zeta_i(\cdot | a_i^n, t_i)$ are uniformly bounded, hence $\exists \alpha > 0$ such that:

$$\int_{[x^*, \infty)} |x^2| \eta_i(x | a_i^n, t_i) \lambda(dx) \leq \alpha, \quad \forall c > 0 \text{ and } \forall n.$$

Then:

$$\begin{aligned}
v^*(x-H) & \int_{|x|>c} \eta_i(x|a_i^n, t_i) \lambda(dx) \\
& \leq (v(0) - v'(0)\underline{\theta}) \int_{|x|>c} \eta_i(x|a_i^n, t_i) \lambda(dx) + v'(0) \frac{1}{c^2} \int_{[x^*, \infty)} |x^2| \eta_i(x|a_i^n, t_i) \lambda(dx) \\
& \leq (v(0) - v'(0)\underline{\theta}) \int_{|x|>c} \eta_i(x|a_i^n, t_i) \lambda(dx) + v'(0)\alpha
\end{aligned}$$

In the proof of proposition 6, it was shown that K is v -tight, so that $\forall \varepsilon > 0, \exists c(\varepsilon) > 0$ such that, $\forall n$:

$$\int_{|x|>c(\varepsilon)} |v(x - s^n(x))| \eta_i(x|a_i^n, t_i) \lambda(dx) \leq \varepsilon$$

Therefore [Page (1987), proposition 3.1] $E_i v(s^n, a_i^n, t_i) \rightarrow E_i v(s, a_i, t_i)$. Since Φ is metrizable for the weak convergence, it follows that $E_i v(\cdot, t_i)$ is continuous on $\Phi \times A_i, \forall t_i \in T_i$.

Finally, assumption (C4-iii) and the measurability of v imply that $E_i v(s, a_i, \cdot)$ is \mathfrak{F}_i -measurable, $\forall a_i \in A_i$. Then $\mathbb{D} \otimes \mathbb{D}_i \otimes \mathfrak{F}_i$ -measurability follows immediately [Castaing & Valadier (1977), lemma III.14]. ■

The principal wants to implement a contract that makes the winning agent choose an action that maximizes the principal's expected utility functional, that is, the principal requests from the winning agent an optimal action that solves:

$$\sup \{ E_i v(id - s, a_i, t_i) : a_i \in B_i(s, t_i) \}$$

Let $v^*(i, s, t_i) = \sup \{ E_i v(id - s, a_i, t_i) : a_i \in B_i(s, t_i) \}$ be the principal's optimal expected utility if the agent i of type t_i wins the auction and is awarded contract s , provided the agent takes the action $a_i \in B_i(s, t_i)$ suggested by the principal.

Given a type profile $t \in T$ and a contract $s \in \Phi$, the vector of agents' optimal expected utilities is $(u^*(1, s, t_1), \dots, u^*(m, s, t_m)) \in \mathfrak{R}^m$ and the principal's vector of optimal expected utilities is $(v^*(1, s, t_1), \dots, v^*(m, s, t_m)) \in \mathfrak{R}^m$.

The possibility of no contracting is modeled by introducing a fictitious agent, labeled $i = 0$, such that $u^*(0, s, t) = r_0(t), \forall (s, t) \in \Phi \times T$, where $r_0(t)$ is the fictitious agent's reservation utility given the type profile. The fictitious agent does not report any type, but if based on the type profile of the actual agents she wins the auction, then she receives her reservation utility. The principal is also supposed to have a reservation payoff $\phi_0(t)$,

$\forall (s,t) \in \Phi \times T$, from no contracting in such a way that $v^*(0,s,t) = \varphi_0(t)$. Now define the agents' utility profile and the principal's utility profile, respectively by:

$$(u^*(0,s,t), u^*(1,s,t_1), \dots, u^*(m,s,t_m)) \in \mathcal{V}^{m+1}$$

$$(v^*(0,s,t), v^*(1,s,t_1), \dots, v^*(m,s,t_m)) \in \mathcal{V}^{m+1}$$

With some abuse of notation, but without any loss of meaning, consider the extended set of agents as $I = \{0, 1, \dots, m\}$, and assume that all the functions depending on I are naturally extended to this extended set of agents. The vectors of utilities profiles above can be described, respectively, by the images of the maps $u^*: I \times \Phi \times T \rightarrow \mathcal{R}^{m+1}$ and $v^*: I \times \Phi \times T \rightarrow \mathcal{R}^{m+1}$.

The following proposition establishes some properties the utilities must satisfy in order to prove that this model fits the general model of Balder (1996).

PROPOSITION 9: *Assume (C1)-(C4). The agents' utilities profile satisfies the following properties: $\forall (s,t) \in \Phi \times T$, $u^*(\cdot, s, t)$ is continuous on I ; $\forall t \in T$, $u^*(\cdot, \cdot, t)$ is upper semicontinuous on $I \times \Phi$; and u^* is $\mathbb{F} \otimes \mathcal{D} \otimes \mathcal{S}$ -measurable. The principal's utility profiles satisfies the following properties: $\forall t \in T$, $v^*(\cdot, \cdot, t)$ is continuous on $I \times \Phi$; and v^* is $\mathbb{F} \otimes \mathcal{D} \otimes \mathcal{S}$ -measurable.*

PROOF: Consider the agents' utilities profile map $u^*: I \times \Phi \times T \rightarrow \mathcal{R}^{m+1}$ and the image of some point in the domain as given by:

$$(u^*(0,s,t), u^*(1,s,t_1), \dots, u^*(m,s,t_m)) \in \mathcal{R}^{m+1}.$$

I will show that each component is continuous on I , $\forall (s,t) \in \Phi \times T$. Actually, this is trivial because I is endowed with the discrete topology (as generated by the discrete metric), and the discrete topology has the property that every function defined on the underlying space I is continuous. Then each component $u^*(\cdot, s, t)$ is continuous on I , so is the map u^* of utilities profiles.

Now fix $t \in T$. Then each $t_i \in T_i$ is fixed. I will prove that each component is upper semicontinuous on $I \times \Phi$. By assumption (C1-ii), the agent i 's utility function is such that, $\forall t_i \in T_i$, $u_i(\cdot, \cdot, t_i): \mathcal{R} \times A_i \rightarrow \mathcal{R}$ is continuous. By the assumptions (C1) and (C4), we have that the conditions for proposition 7 are fulfilled. Then, by proposition 7, the agent i 's best response correspondence $B_i: \Phi \times T_i \rightarrow \mathcal{A}(A_i)$ is nonempty compact valued and $(\mathcal{D} \otimes \mathcal{S}_i, \mathcal{D}_i)$ -measurable, and, $\forall t_i \in T_i$, $B_i(\cdot, t_i)$ is upper semicontinuous on Φ . Then its graph is measurable, closed, and is contained in a compact set, hence it is compact. Therefore, by the Berge's maximum theorem [Aliprantis & Border (1994), theorem 14.30], the value function $u^*(i, s, t_i) = \sup\{E_i u_i(s | a_i, t_i) : a_i \in A_i\}$ is upper semicontinuous on Φ .

This shows that each component is upper semicontinuous on Φ . Since we already know that it is continuous on I , it follows that it is upper semicontinuous on $I \times \Phi$. Then $\forall t \in T$, $u^*(\cdot, \cdot; t)$ is upper semicontinuous on $I \times \Phi$.

Since $u^*(\cdot, \cdot; t)$ is upper semicontinuous on $I \times \Phi$, and \mathfrak{F} -measurable on T , it follows from that u^* is $\mathfrak{F} \otimes \mathfrak{D} \otimes \mathfrak{F}$ -measurable [Castaing & Valadier (1977), lemma III.14].

Now consider the principal's utility profile. By proposition 8, each component is continuous on $\Phi \times A_i$, $\forall t_i \in T_i$, and $\mathfrak{D} \otimes \mathfrak{D}_i \otimes \mathfrak{F}_i$ -measurable. Then $v^*(\cdot, \cdot; t)$ is continuous on $I \times \Phi$, and v^* is $\mathfrak{F} \otimes \mathfrak{D} \otimes \mathfrak{F}$ -measurable. ■

Now we are able to describe the role of individual rationality in Page (1994)'s model.

For each truthfully reported type profile $t \in T$, consider the set $\Pi(t) = \{(i, s) \in I \times \Phi : u^*(i, s, t) \geq r(i, t)\}$. Recall that $K = \text{Meas}(T, \text{Prob}(I \times \Phi))$. The rational contract correspondence $\Gamma: T \rightarrow P(K)$ is given by $\Gamma(t) = \{\xi \in K : \text{supp} \xi(t) \subset \Pi(t), \nu\text{-a.s.}\}$, that is, the rational contract correspondence consists of all probability measures (or mixed contracts) with support contained in $\Pi(t)$. Page (1994) calls it *ex post individual rationality*. According to our taxonomy, this corresponds to the almost sure individual rationality introduced in the general model. Then according to the taxonomy of the general model, Page (1994) works with almost surely individually rational mechanisms, that is, with the set $S(asIR)$. Furthermore, in order to avoid the willingness to not participate in the auction, Page (1994) assumes that some actual agent satisfies individual rationality. This is because the fictitious agent trivially does, but Page (1994) is of course interested in the nontrivial case. This is described by the assumption below:

$$(C5) \exists (i^*, s^*) \in (I \setminus \{0\}) \times \Phi \text{ such that } (i^*, s^*) \in \Pi(t), \forall t \in T.$$

In the next result I prove that assumption (A2) is satisfied, that is, that the rational contract correspondence behaves like in the general model of Balder (1996).

PROPOSITION 10: *Assume that (C1)-(C5) hold. Then the rational correspondence $\Gamma: T \rightarrow P(K)$ satisfies (A2).*

PROOF: Convexity will be shown first. Let $\xi, \gamma \in \Gamma(t)$ and $0 \leq \alpha \leq 1$. Then $\exists N^1 \in \mathfrak{F}$ with $\nu(N^1) = 0$ such that $\text{supp} \xi(t) \subset \Pi(t)$, $\forall t \in T \setminus N^1$, and $\exists N^2 \in \mathfrak{F}$ with $\nu(N^2) = 0$ such that $\text{supp} \gamma(t) \subset \Pi(t)$, $\forall t \in T \setminus N^2$. Let $N^0 = N^1 \cup N^2$. Then it follows that $N^0 \in \mathfrak{F}$, $\nu(N^0) = 0$, and, $\forall t \in T \setminus N^0$,

$\text{supp}\xi(t) \subset \Pi(t)$ and $\text{supp}\gamma(t) \subset \Pi(t)$. Define the measure $\delta(t) = \alpha\xi(t) + (1-\alpha)\gamma(t)$. By proposition 6, K satisfies (A1), hence it is convex. Then the probability $\delta(t)$ is well defined. Moreover, $\forall t \in T \setminus N^c$:

$$\text{supp}\delta(t) = \text{supp}\{\alpha\xi(t) + (1-\alpha)\gamma(t)\} \subset \alpha \text{supp}\xi(t) + (1-\alpha) \text{supp}\gamma(t) \subset \Pi(t).$$

Then $\text{supp}\delta(t) \subset \Pi(t)$, $\forall t \in T \setminus N^c$, i.e., v-a.s. Then $\delta \in \Gamma(t)$, so $\Gamma(t)$ is convex.

Now I will show that $\Pi(t)$ is closed. Since we assume (C1)-(C4), we can apply proposition 9. Then, by proposition 9, $u^*(\cdot, t)$ is upper semicontinuous on $I \times \Phi$, $\forall t \in T$. Since the topology on I is discrete, the reservation utility function is trivially continuous on I . Then the set $\Pi(t) \subset I \times \Phi$ is closed in the relative topology inherited from $I \times \Phi$. By assumption (C5), it is nonempty. Since $I \times \Phi$ is compact, the result follows [Aliprantis & Border (1994), theorem 12.17]. ■

Page (1994) assumes that the principal faces the set of all agents, including the fictitious one, as a whole. Specifically, the principal does not think of each agent separately. She offers a contract to the set of agents as a whole, allows them to participate in the auction, and finally awards the contract to the winning agent. It allows Page (1994) to generalize the model to the multi-agent framework. It is done by redefining the set over which the principal calculates expected utilities, and by assuming that each agent calculates expected utilities over the set of the remaining agents (this is the nature of ex post individual rationality) and the set of contract payoffs. To see this, consider a direct auction mechanism $f \in K$. Then the utility of agent i , given the others report truthfully, is the functional $U_i : T_i \times K \rightarrow \mathcal{R}$ defined by:

$$U_i(t_i, f) = \int_{T_{-i} \times \Phi} u^*(i, s, (t_i, t_{-i})) f(d(-i, s) | (t_i, t_{-i}))$$

Now I can show that assumption (A3) is satisfied for each individual agent.

PROPOSITION 11: *Each $U_i : T_i \times K \rightarrow \mathcal{R}$ satisfies (A3).*

PROOF: Fix $t \in T$, so each $t_i \in T_i$ is fixed. Consider $f \in \Gamma(t)$. Then in particular $f(t)$ is a probability measure on $I \times \Phi$. The functional $U_i(t_i, \cdot) : \Gamma(t) \rightarrow \mathcal{R}$ defined by:

$$U_i(t_i, f) = \int_{T_{-i} \times \Phi} u^*(i, s, (t_i, t_{-i})) f(d(-i, s) | (t_i, t_{-i}))$$

is clearly a linear functional on a subset of the space of probability measures on compacta. Hence in particular it is affine and continuous on $\Gamma(t)$ in the weak topology. Then assumption (A3) is immediately satisfied. ■

In Page (1994), each agent has an individual misspecification correspondence $M_i: T \rightarrow P(T_i)$, which is given by $M_i(t) = \{(t'_i, t_{-i}) : t'_i \in T_i\}$. Notice that $M_i(t)$ is just the projection of T onto its i -th component. The interpretation of this is that, given the type profile t , agent i is allowed to misreport only her own type. Finally, set the misspecification correspondence $M: T \rightarrow P(T)$ as $M(t) = \times_{i=1}^m M_i(t)$.

A mechanism f is said to be dominant strategy almost surely incentive compatible for agent i (i -asIC) if $U_i(t_i, s) \geq U_i(t'_i, s)$, $\forall t'_i \in M_i(t)$, v-a.s. The mechanism f is *dominant strategy almost surely incentive compatible* if it is (i -asIC) $\forall i \in I$. Let $S(i\text{-asIC})$ be the set of all dominant strategy almost surely incentive compatible mechanisms for an agent i .

In Page (1994), the kind of incentive compatibility he works with is the almost sure incentive compatibility. In other words, the set of almost surely incentive compatible mechanisms is given by $S(\text{asIC}) = \cap_{i=1}^m S(i\text{-asIC})$. The set of contracts is then given by the menu of quasi-feasible contracts $S(\text{asIR}) \cap S(\text{asIC})$.

The principal's utility under the mixed contract $f(t)$ is then the functional $V: T \times K \rightarrow [-\infty, \infty)$ given by:

$$V(t, f(t)) = \int_I \int_{\Phi} v^*(i, s, t) f(t)(d(i, s))$$

The last assumption to be satisfied is (A4). In what follows I show it.

PROPOSITION 12: *The principal's utility $V: T \times K \rightarrow [-\infty, \infty)$ satisfies (A4).*

PROOF: By proposition 9, $v^*(\cdot, \cdot, t)$ is continuous on $I \times \Phi$, $\forall t \in T$, and v^* is $\mathbb{F} \otimes \mathbb{D} \otimes \mathfrak{S}$ -measurable. Then $\Psi(t) = V(t, f(t))$ is clearly \mathfrak{S} -measurable. Fix $t \in T$. Since $f(t)$ is a probability measure, $\forall f \in \Gamma(t)$, then clearly $V(t, \cdot)$ is linear on $\Gamma(t)$. In particular it is (weakly) concave. Since it is linear, it follows that it is continuous. In particular, it is upper semicontinuous.

Let $f \in S(\text{asIR}) \cap S(\text{asIC})$. Since $I \times \Phi$ is compact and $v^*(\cdot, \cdot, t)$ is continuous, $\forall t \in T$, it follows that $v^*(\cdot, \cdot, t)$ is uniformly continuous and uniformly bounded by a constant, say H . Then:

$$V(t, f(t)) = \int_I \int_{\Phi} v^*(i, s, t) f(t)(d(i, s)) \leq \int_I \int_{\Phi} H f(t)(d(i, s)) = H < \infty$$

Then V is integrably bounded on $f \in S(\text{asIR}) \cap S(\text{asIC})$ with respect to T . ■

Then the objective function of the principal is finally:

$$I_V(f) = \int_T V(t, f(t)) \nu(dt)$$

Hence in Page (1994)'s model, the principal solves the quasi-maximization problem:

$$(asP) \quad \sup_{f \in \mathcal{S}_{(asIR)} \cap \mathcal{S}_{(asIC)}} I_V(f)$$

Since all the entities involved in the model satisfy the assumptions of the general model of Balder (1996), it follows from the quasi-existence theorem that a solution exists.

5. Mixed contracts with one agent

In this section, I present the model of Page (1991). This is exactly like the model of optimal contracts in the principal-multi-agent setting presented in the previous section. The only fundamental difference is the fact that now there is only one agent, so that the set of agents is a singleton $I = \{i^*\}$. Then it is not necessary to repeat the set-up neither the propositions that show how the model fits the general model of Balder (1996).

In this case the menu of contracts is given by $K = \text{Meas}(T, \text{Prob}(\Phi))$, i.e., a contract mechanism f is in the menu of (mixed) contracts if it assigns to each type t a probability measure $f(t)$ on the set of contract payoffs, so that the contract is selected from Φ according to the probability measure $f(t)$ or, in a more understandable notation, $f(\cdot | t)$, which means a probability measure that, conditional on the revealed type, describes the distribution of payoffs from the agent's action.

It is important to understand what the set Φ is in Page (1991)'s model. Given a probability space $(\Omega, \mathcal{F}, \mu)$ that describe the uncertainty about the states of the world, where Ω is a separable metric space, Page (1991) considers the set $\text{Meas}(\Omega, \mathcal{R})$ of all Borel-measurable real-valued functions on the probability space. Then Φ is the set of all non-redundant contracts in $\text{Meas}(\Omega, \mathcal{R})$. Recall that two contracts are redundant whenever they pay off equally almost everywhere [see assumption (C3) and in section 3]. In addition, Page (1991) considers a Borel-measurable function $\pi: \Omega \rightarrow \mathcal{R}$ that describes the monetary outcome in each state of nature. So his model is also a model of state-contingent contracts.

For the sake of completeness, notice that since the agent does not randomize among other players' types, her utility is simply:

$$U(t, f(t)) = \int_{\Phi} u^*(s, t) f(ds | t).$$

Similarly for the principal:

$$V(t, f(t)) = \int_{\Phi} v^*(s, t) f(ds | t).$$

Here the integrands are given by:

$$u^*(s, t) = \int_{\Omega} u(s(\omega), t) f(ds | t) \mu(d\omega)$$

$$v^*(s, t) = \int_{\Omega} v(\pi(\omega) - s(\omega), t) f(ds | t) \mu(d\omega)$$

where $s(\omega)$ is the agent's contract payoff (her wage) and $\pi(\omega) - s(\omega)$ is the principal's profit.

Another point is that in Page (1991), the incentive compatibility and the individual rationality must hold everywhere, hence Page (1991) works with a menu of feasible contracts as given by $\mathcal{S}(\mathbb{R}) \cap \mathcal{S}(\mathbb{I}C)$. Moreover, the nonemptiness of the set $\mathcal{S}(\mathbb{R})$ is introduced as an assumption. However, as Balder (1996) shows, this assumption holds whenever the reservation utility function is assumed to be measurable and the rational contract correspondence is assumed to be defined via such a reservation utility function (see proposition 1). This is exactly the case in Page (1991).

Moreover, the misspecification correspondence is trivial, i.e., $M(t) = T$, so that there is no exogenous *a priori* restriction on the agent's right to misreport. As before, the only restrictions come *a posteriori* from the incentive compatibility conditions, and this is internal to the principal-agent model.

Therefore, in Page (1991) the principal faces a menu of feasible contracts as her constraint set and finally solves a maximization problem:

$$(P) \quad \sup_{f \in \mathcal{S}(\mathbb{R}) \cap \mathcal{S}(\mathbb{I}C)} I_V(f)$$

Since the model in Page (1991) is just a particular case of Page (1994) with one agent and since all the assumptions are the same, it follows that a solution exists by the existence theorem.

6. Generating menus of contracts

6.1. Introduction

In this section I show some contract sets (precisely, sets of payoff functions of underlying contracts) that might generate menus of contracts that satisfy assumption (A1). Essentially, it is required that these sets satisfy some kind of compactness. The first example is the equicontinuous contract set. The second example is the set of composition contracts. The third one is the set of nondecreasing payoff functions. The fourth one is the set of tight contracts. The fifth one is the set of indexed contracts. The last example is the set of integrably bounded contracts.

6.2 Equicontinuous contract sets

Let $([0,1], \mathcal{B}, \lambda)$ be a probability space, where λ is the Lebesgue measure on the Borel- σ -field \mathcal{B} . This space is assumed to describe the uncertainty about the events of the world. So $([0,1], \mathcal{B}, \lambda)$ is the space $(\Omega, \mathcal{F}, \mu)$ of preceding sections. A contract payoff function is any Borel-measurable function $s: [0,1] \rightarrow \mathcal{R}$, so the underlying space of contracts is $E = \text{Meas}([0,1], \mathcal{R})$. Let $\Phi \subset E$ be a nonempty family of contract payoff functions. The set Φ is *equicontinuous* if, $\forall \omega \in [0,1]$ and $\forall \varepsilon > 0$, $\exists \delta > 0$ such that, if $\omega' \in [0,1]$ is such that $|\omega - \omega'| < \delta$

$\omega^1 < \delta$, then $|s(\omega) - s(\omega')| < \varepsilon$, $\forall s \in \Phi$. It follows from the Arzelà-Ascoli theorem [Bruckner, Bruckner & Thomson (1997), theorem 9.62] that if Φ is equicontinuous and uniformly bounded, then Φ is relatively compact. A simple example of equicontinuous set is the set of all functions $s \in E$ such that, for some constants $\gamma > 0$ and $0 < \alpha \leq 1$, $s \in E$ satisfies the inequality $|s(\omega) - s(\omega')| \leq \gamma |\omega - \omega'|^\alpha$. Call this set $\Phi = \text{Hölder}_{\gamma, \alpha}([0, 1], \mathcal{R})$. In the case $\alpha = 1$, $\text{Hölder}_{\gamma, 1}([0, 1], [\underline{\theta}, \bar{\theta}]) = \text{Lip}_\gamma([0, 1], [\underline{\theta}, \bar{\theta}])$ is the set of Lipschitz-functions. The set Φ then characterizes the exogenous kind of payoff functions (or compensation functions) the principal is constrained to. Suppose that the principal is uncertain about the true type (out of a continuum) of the agent, that is, the principal faces an adverse selection problem. Moreover, assume the principal offers a type-contingent mixed contract, that is, for each revealed type, the principal chooses a contract from Φ according to a probability measure conditional on the revealed type. Then the menu of contracts is given by $K = \text{Meas}(T, \text{Prob}(\Phi))$, which satisfies (A1).

6.3 Composition contract sets

Fix a Borel-measurable function $\mathcal{K}: \mathcal{R} \rightarrow \mathcal{R}$ that assigns to each outcome a score. Call \mathcal{K} the *scoring function*. Suppose that each contract $s \in \Phi$ can be decomposed into $s = h \circ \mathcal{K}$, where $h: \mathcal{R} \rightarrow [\underline{\theta}, \bar{\theta}]$ is a Borel-measurable bounded payoff function that assigns to each score a monetary payoff. Assume that $h \in \mathcal{N}$, where $\mathcal{N} \subset E$. If \mathcal{N} is sequentially compact and uniformly bounded, then Φ is sequentially compact. For example, suppose that the scoring function is given by $\mathcal{K}(x) = \sum_{n=1}^N \frac{n}{n+1} \cdot \chi_{[n-1, n)}(x) + \chi_{[N, \infty)}(x)$, where χ is the characteristic function and $N > 0$ is some arbitrary positive integer. The interpretation of this scoring function is very simple. If the agent produces a nonpositive outcome, then the score of her production is $s(x) = 0$. If she produces $x \in [0, 1)$, then the score of her production is $s(x) = 1/2$. If she produces an outcome $x \in [1, 2)$, then her score is $s(x) = 2/3$. If she produces an outcome $x \in [2, 3)$, then the score of her production is $s(x) = 3/4$, etc. In general, if she produces an outcome $x \in [n, n+1)$, for $n < N-1$, then the score of her production is $s(x) = n/(n+1)$. If she produces any outcome $x \geq N$, then the score of her production is $s(x) = 1$. The greater her production, the greater her score. Now assume that \mathcal{N} is the set $\text{Hölder}_{\gamma, \alpha}([0, 1], [\underline{\theta}, \bar{\theta}])$. Then \mathcal{N} is equicontinuous (see example 6.2). Therefore $\Phi = \{s \in \mathcal{M}([0, 1], \mathcal{R}) \mid s = h \circ \mathcal{K}, \forall h \in \mathcal{N}\}$ is equicontinuous and uniformly bounded, hence it is sequentially compact. This set of compensation functions is the set of example 6.2 filtrated by the scoring function. Again, in the case of mixed contracts with adverse selection, the menu of contracts is $K = \text{Meas}(T, \text{Prob}(\Phi))$.

6.4 Nondecreasing payoff functions

Suppose the principal is constrained to offer a compensation scheme that is bounded and nondecreasing. That is, after observing the outcome level produced by the agent, the principal has to pay a compensation that satisfies the following property: the greater the production level, the greater the compensation scheme, and in addition, whatever the production level, the compensation scheme is bounded by some interval $[\underline{\theta}, \bar{\theta}]$. In this case, the set of payoff functions can be described by $\Phi = \text{Meas}^\uparrow(\mathcal{R}, [\underline{\theta}, \bar{\theta}])$, i.e., the set of all measurable nondecreasing functions with values on $[\underline{\theta}, \bar{\theta}]$. Suppose the principal faces an adverse selection problem, but the contracts are not mixed, that is, the principal offers a pure contract that is a type-dependent compensation scheme. In other words, after observing the revealed type, the principal offers the optimal compensation scheme to the agent that is nondecreasing on the agent's output. In this case, $K = \text{Meas}(T, \Phi)$. If the contracts are mixed, then $K = \mathcal{M}(T, \text{Prob}(\Phi))$.

6.5 Tight contracts

Let Φ be a separable metric space and let \mathcal{A} be a family of finite (Borel) measures on Φ . The family \mathcal{A} is *tight* if for each $0 < \varepsilon < 1$ there exists a compact set $\Theta \subset \Phi$ such that $\xi(\Theta) \geq \xi(\Phi) - \varepsilon$, $\forall \xi \in \mathcal{A}$. Every tight family of measures is relatively compact. This set of contracts appears in Page (1989, 1992 and 1994) and Balder (1995). Elements of $\Phi = \text{Prob}(X)$ are said mixed contracts, in the sense that they are probability measures on the space of payoff functions, and that the principal chooses a contract selection mechanism according to these probability measures. A pure contract is an element $x \in X$. Obviously a pure contract x can be identified with measures concentrated on the point x (i.e., Dirac measures δ_x on x). This is because X can be embedded in $\text{Prob}(X)$ via $x \rightarrow \delta_x$. Moreover, it can be shown that every mixed contract can be arbitrarily approximated by convex combinations of pure contracts. This is because of the density theorem [Aliprantis & Border (1994), theorem 12.9], the set of probabilities with finite support (pure contracts) is dense in $\text{Prob}(X)$.

6.6 Indexed contract sets

Let (\mathcal{Y}, δ) be a compact metric space with metric δ . The set \mathcal{Y} is the index set. Let (Ω, \wp, μ) be a probability space of the events of the world. Suppose that the principal is constrained to offer compensation schemes of the form $s: \Omega \times \mathcal{Y} \rightarrow [\underline{\theta}, \bar{\theta}]$, where, $\forall \omega \in \Omega$, $s(\omega, \cdot)$ is (sequentially) continuous on \mathcal{Y} , and, $\forall y \in \mathcal{Y}$, $s(\cdot, y)$ is \wp -measurable. Such a compensation scheme is called an indexed contract. Let $\Phi = \{f(\cdot, y) / y \in \mathcal{Y}\}$ be the collection of all such indexed contracts. Then Φ is sequentially compact. Therefore, thinking again about mixed contracts with adverse selection, the set $K = \text{Meas}(T, \text{Prob}(\Phi))$ satisfies (A1). As Page (1987) points out, many sets of contracts consisting of piecewise

linear contracts can be reformulated as indexed contract sets with compact index sets. The sets consisting of combinations of finitely many contracts can be also viewed as indexed contract sets with compact index sets.

6.7 Integrably bounded contracts

In section 3 the set of contracts was modeled as the set $S^1(X)$ of integrable selection from the correspondence $X:\Omega\rightarrow B$ [Balder & Yannelis (1993)]. Under suitable conditions on the correspondence X , the set $S^1(X)$ is relatively weakly compact. The fundamental condition is that X be integrably bounded [Yannelis (1991), theorem 3.1, and Diestel & Uhl (1977), theorem III.2.15, p. 76]. This is exactly the statement of condition (B1) of section 3 on state contingent contracts. This condition mimics the so-called Dunford theorem¹ [Diestel & Uhl (1977), theorem III.2.15, p. 76] Therefore, whenever the set of contracts is described as the set of integrable selections from a suitable correspondence satisfying these conditions, then $K = S^1(X)$ satisfies (A1).

7. The model machine

The above examples suggest that we can find an algorithm to construct menus of contracts that fit the infinite-dimensional approach. Let us call this algorithm the *model machine*:

STEP 1: (a) Choose a collection Φ of compensation schemes (payoff functions) that is relatively compact (a sufficient condition for that is: Φ is equicontinuous and uniformly bounded); or: (b) choose a correspondence $X:\Omega\rightarrow B$ that satisfies the conditions of Dunford theorem [Yannelis (1991), theorem 3.1, or Diestel & Uhl (1977), theorem III.2.15, p. 76].

STEP 2: If you chose (a) in step 1 and want to work with mixed contracts under adverse selection, then use $K=Meas(T, Prob(\Phi))$; if you chose (a) in step 1 and want to work with pure contracts under adverse selection, then use $K=Meas(T, \Phi)$. If you chose (b) in step 1 and want to work with contingent contracts, then use $K = S^1(X)$.

STEP 3: Choose the kind of menu of contracts you want: (c) quasi-feasible or (d) feasible. In either case, a contract selection mechanism is a function $f:T\rightarrow K$.

STEP 4: Make sure the utilities of the principal and of the agent satisfy assumptions (A3) and (A4).

STEP 5: If you chose (c) in step 3, then you can use the quasi-existence theorem; if you chose (d) in step (3), then you can use the existence theorem.

¹ As pointed out by Yannelis (1991, p. 7), this condition is known in the literature of economic theory as Diestel's theorem on weak compactness in the space of Bochner-integrable functions. However, as pointed out by Diestel & Uhl (1977, p. 101), the origin of this kind of result, though less general, is due to Dunford.

For each model we have in mind, we have to consider the adequate set Φ or correspondence X , provided they have not been used before in the infinite-dimensional approach to the principal-agent model. It can be easily verified that the infinite-dimensional approach as summarized in the model machine has been used in many economic problems other than the principal-agent problem. For instance, the infinite-dimensional approach has been also used in Stackelberg games with incomplete information [Page (1989)], Stackelberg games with many followers [Page (1992b)], market games with differential information [Page (1992c)], screening problems with moral hazard [Page (1992a)], existence of optimal auctions [Page (1998)], etc.

So if we think of some specific economic problem, we just have to figure out which kind of set Φ or correspondence X describes better the payoff functions. Then a model can be made if we just follow the model machine.

8. Conclusion

Balder (1996) provided a general model to describe the current results about existence of optimal contracts with moral hazard and adverse selection under continuum of constraints. His approach has been named the infinite dimensional approach in this survey and presented in sections 1 and 2. Though intuitively obvious that his paper is more general than the other papers summarized here, as far as I am concerned no survey exists about the infinite dimensional approach applied exclusively to the principal-agent problems. Therefore this survey seems to be a first effort into this direction, though it is obviously not complete.

In general a survey is chronosynoptical in the sense that it presents the results about some topic in a chronological way. This survey however is a little bit different. I have preferred to present the general model first and afterwards to show how the remaining models fit into the general one. This method provides an overview as any other method, but it has the advantage of clarifying the exact differences between particular models by taking the general model as a primitive.

The model machine in section 7 is a useful method to construct principal-agent models within the infinite dimensional approach. It shows clearly that the starting point of all the models is a compact set of functions Φ or an integrably bounded, convex-compact-valued correspondence X . Section 6 provides some examples of such spaces that generate menus of contracts satisfying assumption (A1) of the general model. Either author chooses the desired kind of feasibility: almost everywhere or everywhere. Then all of them follow the procedures described in the model machine. So any one can do the same, provided the starting point (Φ or X) describes a specific economic situation not modeled before.

In order to illustrate the results consider the following example:

EXAMPLE: Suppose that a contract is such that it pays off a stream of wages to an infinitely lived agent. The contract decision is made once and there is no renegotiation. Then $\Phi = \ell_\infty$, the space of norm-bounded sequences, is the set of potential payoffs. Assume that it is Riesz-paired with its norm-dual ℓ_∞' . Suppose the payoffs are nonnegative each period and are order bounded from above by some stream $\alpha = (\alpha_1, \alpha_2, \dots)$ of payoffs such that $\lim_{n \rightarrow \infty} \alpha_n = 0$. The sequence of payoffs can be interpreted as the sequence of present values corresponding to each period. Then the condition above reflects myopic preferences over future monetary payoffs. The payoffs are then constrained to the order interval $\Phi = [0, \alpha]$. The order interval is weakly compact. In fact it is norm-compact. The agent has type space $T = [0, 1]$, where the type is interpreted as the agent's discount factor. Her utility $U: T \times K \rightarrow [0, \infty]$ is defined by:

$$U(t, \theta) = \sum_{n=1}^{\infty} t^n \theta_n$$

Suppose that the mispecification correspondence M is constant and that it is defined by $M(t) = [0, 1 - \varepsilon]$, where $\varepsilon \approx 0$ is positive and sufficiently small. This is an exogenous condition that keeps the agent from getting infinite utility. The reservation utility function is normalized to $r(t) \equiv 0$. A contract selection mechanism is a measurable function $f: T \rightarrow K$ constrained to be a function of the form $f(t) = \{(1/t)_n \wedge \alpha_n\}$, where $\{(1/t)_n \wedge \alpha_n\}$ is the sequence whose entry n is $(1/t)_n \wedge \alpha_n$. Then $U(t, f(t)) = \sum_{n=1}^{\infty} t^{n-1} \wedge \alpha_n$. The individual rationality is trivially satisfied. The incentive compatibility is:

$$U(t, f(t)) = \sum_{n=1}^{\infty} t^{n-1} \wedge \alpha_n \geq U(t, f(t')) = \sum_{n=1}^{\infty} t^n (1/t' \wedge \alpha_n), \quad \forall t' \in [0, 1 - \varepsilon].$$

This condition holds if and only if $t' \geq t$, $\forall t' \in [0, 1 - \varepsilon]$. The principal's utility is a function $V: T \times K \rightarrow [-\infty, \infty)$ given by $V(t, f(t)) = t f(t)$. Therefore the principal's problem is:

$$\begin{aligned} & \sup \int_{[0, 1]} t \{(1/t)_n \wedge \alpha_n\} \lambda(dt) \\ & \text{subject to } t' \geq t, \quad \forall t' \in [0, 1 - \varepsilon]. \end{aligned}$$

Therefore the optimal contract is the constant sequence $f(t) = \alpha$. This means that the contract pays off the upper bound every period. ■

I present here a synoptic table of the models surveyed in the preceding sections together with some other models that have not been surveyed. This survey was concerned only with principal-agent models, but the table also indicates some different applications that follow the same infinite-dimensional approach.

Synoptic Table

<i>Author</i>	<i>Menu of contracts</i>	<i>Adverse selection</i>	<i>Moral hazard</i>	<i>Kind of feasibility</i>	<i>Specific features</i>
Balder & Yannelis (1993)	$Meas(T, S'(X))$	yes	yes	feasibility	state-contingent contracts
Page (1994)	$Meas(T, Prob(I \times \Phi))$	yes	yes	quasi-feasibility	mixed contracts with many agents
Page (1991)	$Meas(T, Prob(\Phi))$	yes	yes	feasibility	mixed contracts with one agent
Page (1987)	$K \subset Meas(\Omega, [a, b])$, K sequentially compact	no	yes	feasibility	Continua of actions and monetary outcomes
Page (1989)	$Meas(T, Prob(\Phi))$	yes	yes	feasibility	Stackelberg games with incomplete information
Page (1998)	$Meas(T, Prob(I \times \Phi))$	yes	yes	quasi-feasibility	auction with many agents
Page (1992)	$Meas(T, Prob(\Phi))$	yes	yes	Quasi-feasibility	Bayesian Stackelberg games

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