Universidade de Brasília
Departamento de Economia

Série Textos para Discussão

The Fifth Consumer's Surplus: An Extension of the Concept of Marshallian Surplus to Preferences with Non-Null Income Effects

Cássia Helena Marchon
Universidade de Brasília

André Luís Rossi de Oliveira
Universidade de Brasília

Texto nº 287
Brasília, Abril de 2003

Department of Economics Working Paper 287
University of Brasilia, April 2003

UNIVERSIDADE DE BRASÍLIA
DEPARTAMENTO DE ECONOMIA

TEXTO PARA DISCUSSÃO Nº 287

The Fifth Consumer's Surplus: An Extension of the Concept of Marshallian Surplus to Preferences with Non-Null Income Effects

Cássia Helena Marchon
Universidade de Brasília

e

André Luís Rossi de Oliveira
Universidade de Brasília

Brasília, 18 de abril de 2003

UNIVERSIDADE DE BRASÍLIA
DEPARTAMENTO DE ECONOMIA
Campus Universitário Darcy Ribeiro
Instituto Central de Ciências
Caixa Postal 04302, 70910-900 Brasília, DF, Brasil
Tel.: (55-61) 3072498, 2723548
Fax: (55-61) 3402311
E-mail: econ@unb.br
http://www.unb.br/ih/eco

The Fifth Consumer's surplus: An Extension of the Concept of Marshallian Surplus to Preferences with Non-Null Income Effects

Cassia Helena Marchon[1]

André Luís Rossi de Oliveira[2]

The argument usually used to justify the use of the variation of the Marshallian surplus as a measure of welfare change can only be applied under two assumptions. The first one is that preferences are such that income effects are null with respect to the good whose price has changed. The second is that the optimal demands for the other goods are not affected by the price change. The Marshallian surplus is meaningless when these assumptions are not valid. This paper tries to expand the concept of Marshallian surplus to cases where the income effect is not necessarily null, creating a new surplus concept called the extractible surplus. It is shown that the variation in the extractible surplus is equivalent to the compensating variation in the case of a price decrease, and equivalent to the equivalent variation in the case of a price increase, as long as preferences are strictly convex and the demand for the other goods are invariant to the price change.

I. Introduction

According to the concept of cardinal utility, consumer welfare (as well as welfare change) was an absolute value measured in *utils*. Given the limitations of such concept, the theory evolved towards the concept of ordinal utility, according to which the absolute value assumed by a utility function is meaningless and only relative values matter. Under this approach, consumer welfare and welfare changes are not absolute measures, but abstract ones, and as such necessarily involve a certain degree of subjectivity. This means that in principle there is no telling which ones are right and which ones are wrong, they only embody different ways of trying to capture consumer welfare.

[1] University of Brasília, Department of Economics.
[2] University of Brasília, Department of Economics.

One of these measures is the Marshallian surplus, defined as the area below the demand curve and above the market price. Its origin can be traced back to Dupuit [1844], but the concept only became popular after the publications of Marshall's *Principles of Economics* in 1890.

In *Principles of Economics*, Marshall defines the consumer's surplus for a given unit of consumption of the good as the difference between the price the individual is willing to pay to not be deprived of the consumption of the good and the price it actually pays. The price he is willing to pay is that specified by the demand function.

It is important to notice, however, that the argument put forward by Marshall to justify the use of his concept as a measure of the consumer's surplus is valid only for preferences which imply null income effects. This is borne out by the process of construction of the demand curve.

Marshall's definition of consumer's surplus is preceded by the assumption of a constant marginal utility of income. This assumption can be interpreted as a constant marginal utility of income with respect to income and to all prices except one, namely the numeraire good. Under certain conditions this is equivalent to the condition that the income effect is null with respect to all goods except the numeraire, as shown by Samuelson [1942].

If we accept this interpretation of the assumption of constant marginal utility of income, the Marshallian surplus is defined for any good as long as it is not the numeraire good, or, in other words, as long as the income effect for that good is null. On the other hand, since there are preferences which don't satisfy this assumption, it seems like an important and justifiable endeavor to try to extend the applicability of the Marshallian surplus as a measure of welfare to more general preferences which are not associated with null income effects. This is exactly the main objective of this paper.

We will try in this paper to construct a measure of welfare and, consequently, of welfare changes which shares the interpretation of the Marshallian surplus but is applicable to non-null income effect preferences. We will do that through a process of building a demand curve which informs the maximum price an individual is willing to pay for each unit of the good consumed, and we will call it the modified demand. Our welfare measure will be defined as the difference between this maximum price and the market price and will be called the extractible surplus.

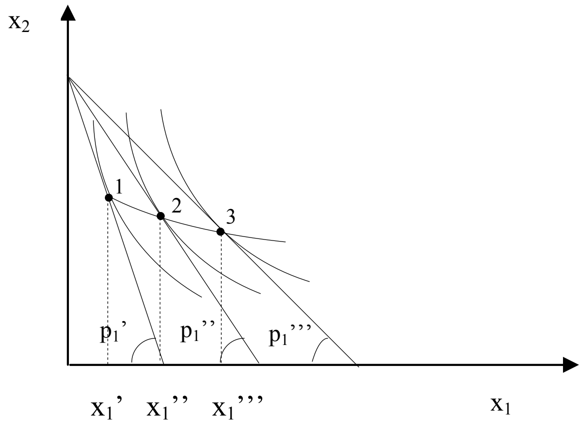
After this Introduction, section 2 describes the procedure to calculate the total extractible surplus for the discrete case. Section 3 will use this new concept to measure changes in the consumer's surplus in the discrete case. In section 4, we will deal with extractible surplus changes in the continuous case. Section 5 will compare the extractible surplus with some Hicksian welfare measures and section 6 concludes the paper.

II. Total Extractible Surplus in the Discrete Case

Although the process of constructing the demand curve is well-known, it is useful for our purposes to reproduce it here. First, we construct a price-consumption curve by varying only the price of one good, namely good 1. Figure 1 below illustrates this procedure for three different price levels. By plotting the resulting pairs of price and optimal consumption of good 1, we obtain the demand curve for this good.

Now let $P_1(.)$ be the inverse demand function for good 1. We can notice from the process of deriving the demand curve that the individual only buys x_1''' units of good 1 because the price of each units is p_1'''. He is not willing to pay $P_1(1)$ for the first unit, $P_1(2)$ for the second unit and so on. He doesn't even have enough income to do that. When he buys x_1''' at price p_1''', he is already spending all of his income. If required to pay a higher price for the consumption of the first units, he would have to spend more than his income, which is impossible.

Figure 1: Price-consumption curve



It is clear then that the demand function is not very helpful when we want to calculate what in fact the individual is willing to pay for each extra unit of consumption. In order to find a new method to help us make such calculations, we make the following assumptions:

1. There are only two goods in the Economy, goods 1 and 2;
2. Good 1 can only be consumed in discrete amounts, i.e., the consumption set is defined by $X = \{(x_1, x_2) : x_1 \in \mathbb{N}, x_2 \in \mathbb{R}_+\}$.
3. The income (*M*) and preferences of the individual are known;
4. Preferences are complete and transitive;
5. Preferences are strongly monotone;
6. Preferences are continuous;
7. If x', x'' and $x''' \in X$, $x' \sim x''$ and $\exists\, \lambda \in (0,1)$ such that $x''' = \lambda x' + (1-\lambda) x''$, then $x''' \succ x' \sim x''$;
8. When the individual is indifferent between two baskets, he always opts for the basket with a larger amount of good 1 (tie-breaking rule); and
9. Good 1 is a normal good.

We will now derive the modified demand. The procedure to construct this demand is presented below, and is divided in several steps.

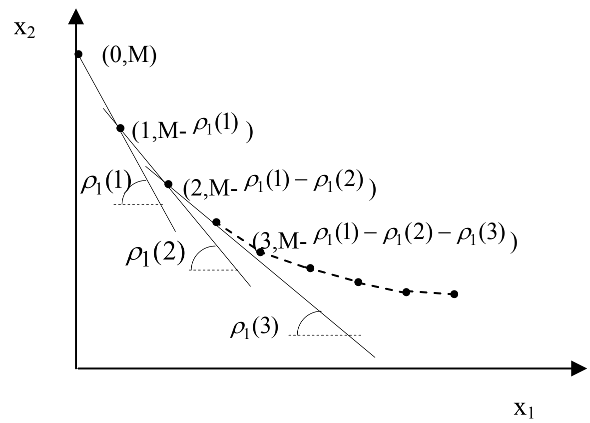
Step 1: We want to find the maximum price an individual is willing to pay for the consumption of the first unit of good 1 when his income is *M*. Let $\rho_1(1)$ be this price.

<u>Graphic Solution:</u>

Notice that the bundle $(0, M)$ always belongs to the budget set, i.e., it can be consumed by the individual independently of the price charged for the first unit of the good.

The slope of the budget line is $-p_1/p_2 = -p_1$, since good 2 is taken to be the numeraire, with price $p_2 = 1$. If $p_1 = +\infty$, the budget line is the segment $[(0,0),(0,M)]$ of the vertical axis. In order to find the maximum price an individual is willing to pay, we reduce the slope of the budget line, with $(0,M)$ as a point of rotation, until the individual's optimal choice is to consume exactly one unit of the good, as illustrated in figure 2. The value taken by the slope at this point is the maximum price the individual is willing to pay for the consumption of the first unit of good 1.

Figure 2: Steps 1, 2 and 3: The budget line is rotated until the individual chooses optimally to consume 1 unit of good 1.



Formal Solution:

It is the same to say that an individual is willing to pay $\rho_1(1)$ to consume the first unit of good 1 and to say that when he is faced with price $\rho_1(1)$ he buys the first unit of the good (this does not exclude the possibility that, at his price, he buys extra units of the good, though). Since this individual only buys the good if it optimal for him to do it, it is necessary that at price $\rho_1(1)$ his optimal consumption of good 1 is greater than or equal to 1. Moreover, $\rho_1(1)$ is equal to the maximum price p_1 such that the optimal consumption of good 1 is greater than or equal to 1. Formally, our problem is to find

$$\rho_1(1) = \max\left\{p_1 : (x_1^*, x_2^*) \succsim (x_1, x_2) \;\; \forall (x_1, x_2) \in B, \text{ where } x_1^* \geq 1 \text{ and } (x_1^*, x_2^*) \in B\right\}, \quad (1)$$

where $B = \{(x_1, x_2) : p_1 x_1 + x_2 \leq M\}$.

Proposition 1: There is a unique solution to problem (1), and it can be found by solving the equation

$$u(0,M) = u(1, M - \rho_1(1)) \tag{2}$$

The proof of this proposition can be found in the appendix.

Let's assume now that the individual paid $\rho_1(1)$ to consume the first unit of good 1. Then his income is reduced to $M - \rho_1(1)$.

Step 2: We want to find the maximum price the individual is willing to pay to consume a second unit of good 1, given that it has already paid $\rho_1(1)$ for the first unit. Let $\rho_1(2)$ be this price.

Notice that, no matter what the price charged for the second unit of good 1, the bundle $(1, M - \rho_1(1))$ is always feasible, i.e, it belongs to the budget set. More precisely, whatever the price of the second unit, $(1, M - \rho_1(1))$ belongs to the boundary of the budget set. This is true because once the individual spends $M - \rho_1(1)$ on the consumption of good 2, he cannot consume more than his initial endowment of good 1, which is 1 unit.

In order to find $\rho_1(2)$, we rotate the budget line obtained in step 1 to the right (or, equivalently, reduce p_1), with $(1, M - \rho_1(1))$ as the center of rotation, until the individual chooses optimally to consume the second unit of good 1. Figure 2 illustrates this procedure. The slope of this new budget line is the maximum price the individual is willing to pay for the second unit of good 1.

A similar argument as the one used in step 1 shows that $\rho_1(2)$ can be found as a solution to the following equation:

$$u(2, M - \rho_1(1) - \rho_1(2)) = u(1, M - \rho_1(1)) \tag{3}$$

Suppose now that the individual paid $\rho_1(2)$ for the second unit of good 1, which leaves him with income equal to $M - \rho_1(1) - \rho_1(2)$.

Step 3: We want to figure out the maximum price the individual would be willing to pay for the third unit of good 1, given the fact that he has already paid $\rho_1(1)$ for the first unit and $\rho_1(2)$ for the second unit. The procedure is also illustrated in figure 2.

A reasoning similar to what was used in steps 1 and 2 shows that $\rho_1(3)$ is the solution to the equation

$$u(3, M - \rho_1(1) - \rho_1(2) - \rho_1(3)) = u(2, M - \rho_1(1) - \rho_1(2)) \tag{4}$$

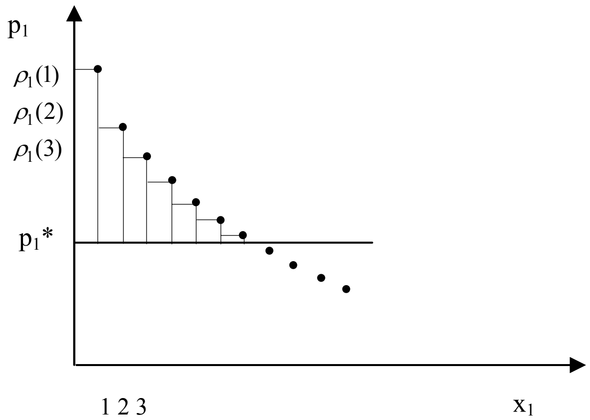
Subsequent steps are obtained analogously.

By plotting points $(1,\rho_1(1)),(2,\rho_1(2)),(3,\rho_1(3)),\ldots,(n,\rho_1(n))$ in a $p_1 \times x_1$ graphic, we obtain what we call the modified demand for good 1, which is represented in figure 3. Since for now we are working with the discrete case, this demand is composed of isolated points. In the continuous case it will be represented by a curve.

The modified demand tells us the maximum price an individual is willing to pay for the consumption of each unit of the good. More precisely, he is willing to pay $\rho_1(1)$ for the first unit, $\rho_1(2)$ for the second unit, $\rho_1(3)$ for the third unit and so on.

We are now in position to derive what we call the extractible surplus. Let p_1^* be the market price of good 1. The individual is willing to pay $\rho_1(1)$ to consume the first unit of the good, but only has to pay p_1^*. Therefore, his surplus with the consumption of the first unit is $\rho_1(1) - p_1^*$ (this value is represented as the area of the first rectangle in figure 3). To consume the second unit of good 1, he is willing to pay $\rho_1(2)$ but only has to pay p_1^*. His surplus with this unit is then $\rho_1(2) - p_1^*$. The procedure is repeated until the difference between what the individual is willing to pay and the market price is zero or negative. The sum of the positive unit surpluses is defined as the total extractible surplus of the consumer when the market price is p_1^*.

Figure 3: Modified Demand and Extractible surplus in the discrete case



The name extractible surplus is due to the fact that it represents the value a monopolist could extract from the consumer if he knew his modified demand. For instance, the monopolist could charge $\rho_1(1)$ for the first unit, $\rho_1(2)$ for the second and so on.

As was the case with the Marshallian surplus, the extractible surplus can be defined as the difference between the individual is willing to pay to guarantee he will not be deprived of its consumption and the price he actually pays.

If we compare the methods of constructing the Walrasian demand and the modified demand, we notice that any difference between them is due the income effect. If this effect is null, then the curves of these two demand functions coincide and the extractible surplus is the same as the Marshallian surplus.

In the next section, we will discuss how to measure changes in the consumer's surplus.

III. Extractible Surplus Variation

In most practical problems, we are interested not in the total surplus but in the surplus variation. For instance, we may be interested in calculating the consumer welfare change associated with an increase in price due to the creation of a tax on consumption or on imports.

Suppose we want to measure the change in the consumer's surplus due to a fall of the price of good 1. Let $\overline{p}_1$ be the initial price and $\underline{p}_1$ the final price, where $\overline{p}_1 > \underline{p}_1$. If we want to calculate the decrease in the Marshallian surplus, we have to compare the area above $\overline{p}_1$ and below the inverse (Walrasian) demand curve with the area below this curve and above $\underline{p}_1$. More exactly, the change in Marshallian surplus is the difference between these two areas.

Let $\overline{x}_1$ be the quantity the individual consumes of good 1 when the price is $\overline{p}_1$ and let $\underline{x}_1$ be the quantity he consumes when the price is $\underline{p}_1$. We can interpret the surplus change in the following way. The individual was willing to pay $\overline{x}_1\overline{p}_1$ for the consumption of $\overline{x}_1$ units of good 1, but after the price fall he was able to buy the same quantity spending only $\overline{x}_1\underline{p}_1$. Hence he obtained a surplus of $\overline{x}_1\overline{p}_1 - \overline{x}_1\underline{p}_1 = \overline{x}_1\left(\overline{p}_1 - \underline{p}_1\right)$. Moreover, in order to consume one extra unit of good 1, the individual was willing to pay $p_1\left(\overline{x}_1 + 1\right)$, where $p_1\left(.\right)$ is the inverse (Walrasian) demand function, but was only required to pay $\underline{p}_1$. Hence the surplus generated by the consumption of this extra unit is $p_1\left(\overline{x}_1 + 1\right) - \underline{p}_1$. To consume another unit of the good, the individual would have paid $p_1\left(\overline{x}_1 + 2\right)$, but only had to pay $\underline{p}_1$, which means that his surplus was equal to $p_1\left(\overline{x}_1 + 2\right) - \underline{p}_1$. Proceeding in this fashion while the surplus generated by an extra unit is positive, we derive the Marshallian surplus change. In the case where the good is consumed in continuous quantities, this change is just the integral of the demand function over the interval $\left[\underline{p}_1, \overline{p}_1\right]$.

Notice that when there is a price raise instead of a fall, the same area can be interpreted as the loss of consumer's surplus due to the price increase.

Once again, the analysis carried out above for the Marshallian surplus is only valid when there are no income effects for good 1. The reason for that is the same we discussed before in the context of the calculation of the total consumer's surplus. If income effects are present, this measure can only be used as an overestimation of the compensating variation and an underestimation of the equivalent variation, when there is a price decrease, and vice-versa.

As a matter of fact, there is another condition that has to be satisfied in order to justify the use of the area described above as a measure of the change in the Marshallian surplus caused by a price decrease. It is necessary that the demands for the other goods do not depend on the price of good 1. To understand why this is true, consider a situation where another good, say good 2, is a substitute for good 1, so that its demand is positively correlated to the price of good 1. If the price of good 1 falls, then the demand curve of good 2 shifts down and to the left, thereby altering the Marshallian surplus in this market.

We will try now to fashion a measure that shares the interpretation given to the Marshallian surplus but that can be applied to more general types of preferences, with non-zero income effects.

Suppose we want to calculate the variation of extractible surplus due to a price fall. Let $\overline{p}_1$ be the initial price and $\underline{p}_1$ the final price, where $\overline{p}_1 > \underline{p}_1$ as before. Moreover, suppose that assumptions 1 to 9 of section 2 are valid.

The change in extractible surplus caused by a price decrease and associated with the units the individual was already consuming before the price change is equal to $\left(\overline{p}_1 - \underline{p}_1\right)\overline{x}_1$, the same as in the case of the Marshallian surplus. The difference between the maximum price he is willing to pay for each extra unit of the good, beyond those he was already consuming, and the new price $\underline{p}_1$ is the change in extractible surplus associated with that extra unit. The sum of all positive surpluses gives us the total change in extractible surplus due to the consumption of extra units of good 1, to which the individual attaches values greater than $\underline{p}_1$.

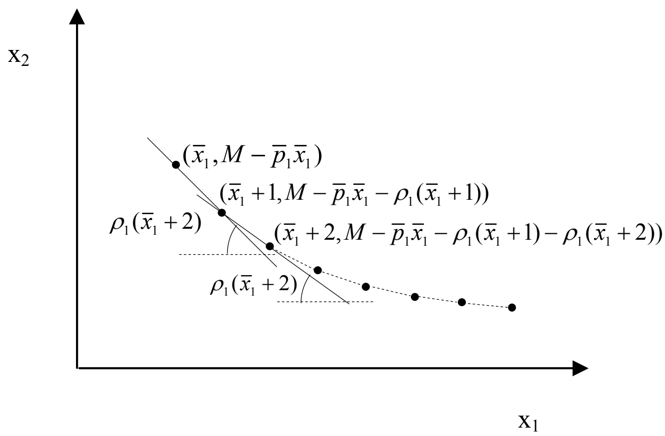
Let's now describe the procedure to calculate the change in extractible surplus in more formal terms. As before, we have to first derive the modified demand for price $\overline{p}_1$. We will do this in steps.

Step 1: We know that the individual is willing to pay $\overline{x}_1\overline{p}_1$ to consume the first $\overline{x}_1$ units of good 1. Now assume that he actually spent $\overline{x}_1\overline{p}_1$ to buy this amount, ending up with income equal to $M - \overline{x}_1\overline{p}_1$.

Step 2: We want to find the maximum price the individual is willing to pay to consume the next unit of good 1 given that he already consumed $\overline{x}_1$ units of the good and has income equal to $M - \overline{x}_1\overline{p}_1$. Let $\rho_1\left(\overline{x}_1 + 1\right)$ be this price.

Notice that the bundle where $x_1 = \overline{x}_1$ and $x_2 = M - \overline{x}_1\overline{p}_1$ is always feasible, independently of the price charged for the next unit of good 1. Thus this bundle belongs to the budget set of the individual. In fact, it belongs to the boundary of this set, for given that the individual consumes $\overline{x}_1$ units of good 1, the maximum amount of good 2 he can buy is $M - \overline{x}_1\overline{p}_1$. In order to find the maximum price he is willing to pay for one more unit of good 1, we reduce its price or, equivalently, the slope of the budget line, using as point of rotation the bundle $\left(\overline{x}_1, M - \overline{x}_1\overline{p}_1\right)$, until the individual's optimal choice is to consume the next unit of good 1 (see figure 4). The slope of this new budget line is $\rho_1\left(\overline{x}_1 + 1\right)$.

Figure 4: Steps 1 to 3: The slope of the budget line is reduced around the optimal choice of the previous step until the optimal choice of the individual is to consume the next unit of good 1.



The price $\rho_1\left(\overline{x}_1 + 1\right)$ can also be obtained as a solution to the equation

$$u\left(\overline{x}_1 + 1, M - \overline{x}_1\overline{p}_1 - \rho_1\left(\overline{x}_1 + 1\right)\right) = u\left(\overline{x}_1, M - \overline{x}_1\overline{p}_1\right) \quad (5)$$

Now suppose the individual has already paid $\rho_1\left(\overline{x}_1 + 1\right)$ for one extra unit of good 1, which leaves him with $\overline{x}_1 + 1$ units of good 1 and income of $M - \overline{x}_1\overline{p}_1 - \rho_1\left(\overline{x}_1 + 1\right)$.

Step 3: We wish to find the maximum price the individual is willing to pay for the consumption of an extra unit of good 1, given that he already has $\overline{x}_1 + 1$ units and his income is given by $M - \overline{x}_1\overline{p}_1 - \rho_1(\overline{x}_1 + 1)$. Let $\rho_1(\overline{x}_1 + 2)$ be this price. Figure 4 illustrates the procedure to get this price, which should be familiar by now.

The price $\rho_1(\overline{x}_1 + 2)$ can be obtained as a solution to the equation

$$u\left(\overline{x}_1 + 2, M - \overline{x}_1\overline{p}_1 - \rho_1(\overline{x}_1 + 1) - \rho_1(\overline{x}_1 + 2)\right) = u\left(\overline{x}_1 + 1, M - \overline{x}_1\overline{p}_1 - \rho_1(\overline{x}_1 + 1)\right) \quad (6)$$

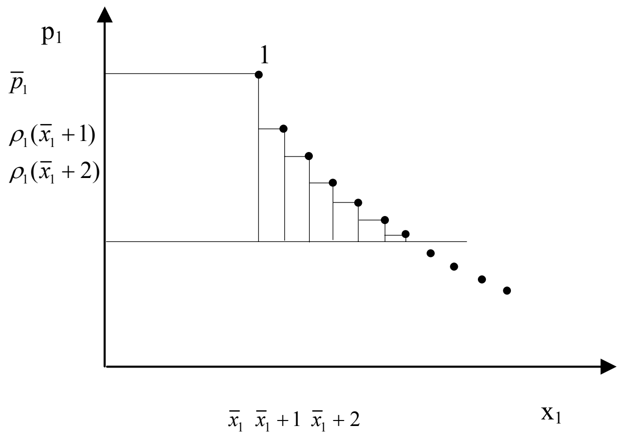
The other steps follow the same reasoning.

If we take the points $(\overline{x}_1, \overline{p}_1), (\overline{x}_1 + 1, \rho_1(\overline{x}_1 + 1)), (\overline{x}_1 + 2, \rho_1(\overline{x}_1 + 2)), \ldots,$ $(\overline{x}_1 + n, \rho_1(\overline{x}_1 + n))$ and plot them on a graphic $x_1 \times p_1$, we obtain the modified demand starting from price $\overline{p}_1$, which is represented in figure 5.

The first point on the modified demand, point 1 on the graphic, tells us that the individual is willing to pay $\overline{p}_1$ to consume the first $\overline{x}_1$ units of good 1. The remaining points tell us the maximum price the individual is willing to pay for each extra unit of good 1.

The individual attributes a value $\overline{x}_1\overline{p}_1$ to the consumption of the first $\overline{x}_1$ units of good 1. However, after the price decrease, he only needs to pay $\underline{p}_1\overline{x}_1$ to buy them, which means that he obtains a surplus of $\overline{x}_1\overline{p}_1 - \overline{x}_1\underline{p}_1 = \overline{x}_1(\overline{p}_1 - \underline{p}_1)$ with the consumption of the first $\overline{x}_1$ units. This surplus is represented by the area of the first rectangle in figure 5. In order to consume the next unit, the individual would be willing to pay $\rho_1(\overline{x}_1 + 1)$, but only has to pay $\underline{p}_1$, and thus his surplus from consuming this unit is $\rho_1(\overline{x}_1 + 1) - \underline{p}_1$. This difference is represented in figure 5 as the area of the second rectangle. We keep on calculating such differences until the surplus generated by an extra unit of consumption is positive. The sum of the areas of all rectangles gives us the variation in the extractible surplus of the consumer caused by the price decrease from $\overline{p}_1$ to $\underline{p}_1$.

Figure 5: Modified demand curve starting from price $\overline{p}_1$ and variation of extractible surplus in the discrete case



In each of the steps above, with the exception of the first, the maximum price the individual is willing to pay for the next unit of good 1 is the price that makes him indifferent between the optimal bundle of the previous step and the optimal bundle where he buys one extra unit of good 1. By transitivity then, we know that the individual is indifferent between all optimal bundles obtained at each step. This means that the modified demand is a segment of the Hicksian demand associated with utility level $u\left(\overline{x}_1, M-\overline{x}_1\overline{p}_1\right)$, more precisely the segment where $p_1 \leq \overline{p}_1$, for the modified demand in this case is not defined for prices above $\overline{p}_1$.

If there is an increase in the price of good 1, from $\underline{p}_1$ to $\overline{p}_1$, the variation in extractible surplus will be the same area we obtained in the case of a price decrease, only now interpreted as the surplus the consumer loses after the price increase.

We asserted before that the variation of the Marshallian surplus would be equal to the trapezoidal area below the demand curve of good 1 only if the demands for other goods in the economy didn't depend on the price of good 1. Otherwise the surplus in the affected market would be different.

Something similar happens with the variation of the extractible surplus. To guarantee that the change in extractible surplus is equal to the sum of the areas of the rectangles below the modified demand for good 1, we need that the modified demand for the other good doesn't depend on the price of good 1. If the maximum prices the individual is willing to pay for the consumption of the extra units of good 2 depend on the price of good 1, then a change in this price will shift the modified demand of good 2, altering the extractible surplus in that market. This means that if we want to calculate the variation in the extractible surplus as the area below the modified demand for good 1, we need to add to the original assumptions the condition that the modified demand for the other good doesn't depend on the price of good 1.

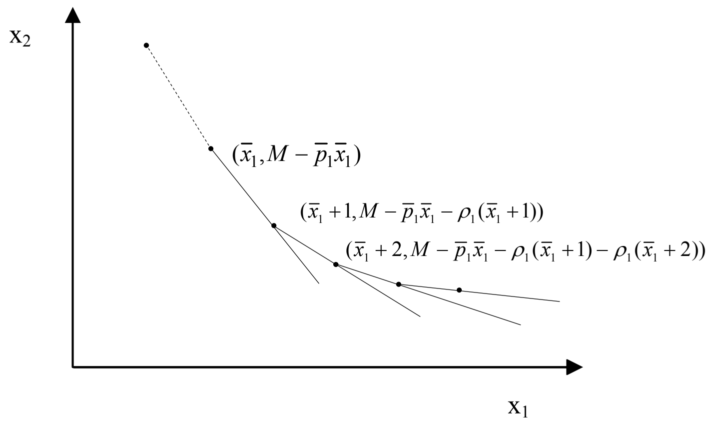
So far we dealt with the extractible surplus in the case where good 1 is consumed in discrete quantities. It is easier to motivate the concept under this hypothesis, but we would like to extend it to the continuous case, and that's what will be done in the next section.

IV. Variation of extractible surplus in the continuous case

In this section we want to calculate the variation in the extractible surplus when good 1 is available in continuous quantities. As before, we will make use of assumptions 1 to 9, and we shall also assume that the modified demand for good 2 does not depend on the price of good 1.

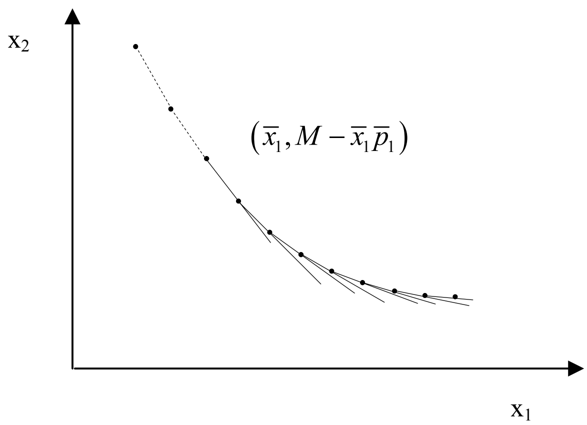
Suppose that the price of good 1 decreases from $\overline{p}_1$ to $\underline{p}_1$ and think about an extremely divisible good, like wheat flour. Assume initially that flour is only available on the shelves in packs of 1 kg. In this case, step 1 of the previous section reduces to calculating how many kg of flour the individual is willing to buy when the price is $\overline{p}_1$. Let $\overline{x}_1$ be this quantity. In step 2, we calculate the maximum price the individual is willing to pay for one extra kg of flour given that he already owns $\overline{x}_1$ kg, and so on. Figure 6 illustrates this procedure.

Figure 6: Bundles that are indifferent to $\left(\overline{x}_1, M - \overline{x}_1\overline{p}_1\right)$ and modified demand when flour is available in 1 kg packs.



Now imagine that flour can be sold in $1/2$ kg packs. Suppose the bundle $(\overline{x}_1, M - \overline{x}_1\overline{p}_1)$ continues to be optimal when the price is $\overline{p}_1$. In this case, step 2 of the construction of the modified demand tells us to calculate the maximum price the individual is willing to pay for an additional $1/2$ kg of flour. We do the same in each subsequent step, as illustrated in figure 7.

Figure 7: Bundles that are indifferent to $(\overline{x}_1, M - \overline{x}_1\overline{p}_1)$ and modified demand when flour is available in $1/2$ kg packs.



We can reduce even further the account unit to 250g. In this case, in each step we calculate the maximum price the individual is willing to pay for an additional 250g of flour. Figure 8 shows how this procedure works graphically.

Figure 8: Bundles that are indifferent to $(\overline{x}_1, M - \overline{x}_1\overline{p}_1)$ and modified demand when flour is available in 250g packs.

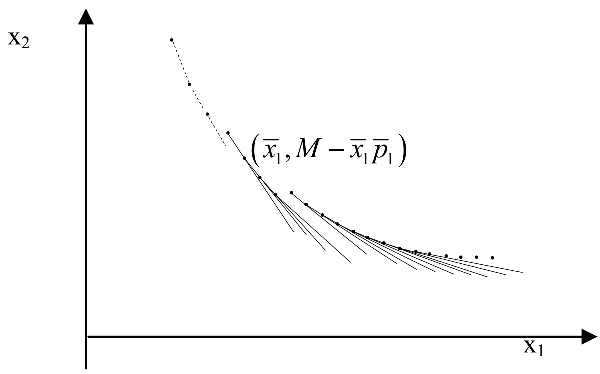
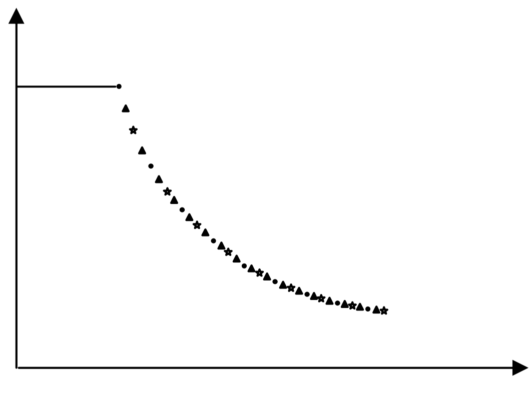


Figure 9 plots together the modified demands for the three cases discussed above. Points represent the modified demand when flour is available in 1 kg packs. Points and asterisks represent it when flour is available in $1/2$ kg packs, and points, asterisks and triangles represent it when flour can be bought in 250g packs.

Figure 9: Modified demands when flour is available in 1kg, $1/2$ kg and 250g packs.



Notice that, no matter how small the account unit, the bundles obtained in each step all belong to the same indifference set. Therefore, the modified demand is equivalent to the Hicksian demand associated to the utility level $u\left(\overline{x}_1, M-\overline{x}_1\overline{p}_1\right)$ for $p_1 \leq \overline{p}_1$.

Notice also that, as the account unit decreases, the modified demand or, equivalently, the Hicksian demand, gets closer to becoming a curve. In the limit, when flour can be consumed in infinitesimal units, the modified demand will be a curve, as shown in the appendix.

By analogy to the discrete case, the variation of the extractible surplus is equal to the integral of the modified demand over the interval $\left[\underline{p}_1, \overline{p}_1\right]$, or equivalently, of the Hicksian demand associated with utility level $u\left(\overline{x}_1, M-\overline{x}_1\overline{p}_1\right)$ over the same interval.

In the case of a price decrease, this integral of the modified demand is exactly the compensating variation. As we know, the compensating variation when the price of good 1 decreases from $\overline{p}_1$ to $\underline{p}_1$ measures how much the income of the individual would have to fall so that, after the price decrease, he gets the same utility as before the price change. We also know that the compensating variation can be calculated as the integral of the Hicksian demand curve associated with the initial utility $u(\overline{x}_1, M - \overline{x}_1\overline{p}_1)$ over $[\underline{p}_1, \overline{p}_1]$.

When there is a price increase, the variation of the extractible surplus is equal to the equivalent variation, which in this case can be interpreted as the reduction in income that would be equivalent in terms of utility change to the price increase. The equivalent variation can be calculated as the integral over $[\underline{p}_1, \overline{p}_1]$ of the Hicksian demand associated with the final utility $u(\overline{x}_1, M - \overline{x}_1\overline{p}_1)$.

In summary, the variation of the extractible surplus is equivalent to the compensating variation when the price goes down and to the equivalent variation when the price goes up.

It is important to keep in mind, though, the conditions under which these equivalences are valid. All the nine assumptions of section 2 plus the condition that the demand for good 2 doesn't depend on the price of good 1 have to be satisfied. Some of them are not necessary for the calculation of the equivalent and compensating variation measures, like the latter and strict convexity.

Next section is dedicated to a more thorough comparison of the notion of extractible surplus and the Hicksian welfare measures, providing an explanation for the title of this paper.

V. Hicks and Extractible Surplus

There are many seminal contributions by Hicks to the theory of the consumer's surplus and welfare measures[3]. One of the most important is *Value and Capital* [1939], where he introduced the notion of compensating variation as a measure of total surplus. In Hicks [1943] we can find his four criteria to calculate the variation of the consumer's surplus, two of them being the compensating and the equivalent variations.

Someone familiar with Hicks' work may question the novelty of the concept we are proposing here. More precisely, we need to answer the question of how it differs

[3] See, for instance, Hicks [1941,1942,1945,1956].

from the compensating and equivalent variations. In order to do that, we will compare Hicks' measures first with the total extractible surplus, and then with the variation in the extractible surplus when the price of only one good is altered.

The total extractible surplus doesn't contribute anything new if compared to the compensating variation as it was defined by Hicks in *Value and Capital*. The compensating variation is defined there as the value that, if extracted from the individual, would leave him with the same utility as what he would get if the market for that good closed or, equivalently, if his consumption of that good was equal to zero.

When the comparison is between the variation of extractible surplus and the compensating variation though, our measure takes a different route vis-à-vis Hicks'. Loyal to Marshall's criterion, we persist in calculating the maximum price the individual is willing to pay for each extra unit of the good. Hicks on the other hand, describes the variation on the consumer's surplus, named by him as the quantity-compensating, and to be measured as the area under what later became known as the Hicksian demand, in the following terms: "Suppose that successive units of the commodity are supplied to the consumer on the following plan. First of all, he is allowed to acquire *HP* units at the price *OH*. At this price he would buy no more units, so in order to induce him to purchase the next unit the price is dropped, but no more than is necessary to induce him to buy that next unit. Then in order to induce him to purchase the unit after that the price is dropped again, but again no more than is necessary. And so on, until the price has been dropped to *Oh*, after which there are no further reductions." In this text, which can be found in Hicks [1943], *HP* is the quantity the individual consumes of the good before the price drops, *OH* is the initial price and *Oh* is the final price.

Although Hicks is aware of the dual interpretation of his compensating variation under a price decrease, he is not worried about being faithful to the Marshall-Dupuit criterion. When he fashioned his measures, he was in fact creating new criteria. For instance, his justification for using the compensating variation as a measure of consumer welfare change is that it represents the income variation necessary to compensate the individual for the price change, so that after the price change he achieves the same utility level as before. If Hicks were interested in extending the Marshall-Dupuit criterion to cases where the income effect is not zero, he would have noticed that the Hicksian demand curves for the other goods would possibly shift as a result of the price change. This in turn would alter the maximum price the individual was willing to pay for the consumption of extra units of the other goods, that is to say his surpluses in those markets. He would then have either to account for such surplus changes or introduce an assumption of independence of the Hicksian demand in one market with respect to price changes in other markets, by no means an insignificant condition. Hicks doesn't take this dual interpretation any further. In fact, his comparison is restricted to the passage above.

In the same fashion, it is possible for the variation of the Marshallian surplus not to be equal either to the compensating variation or to the equivalent variation even if the income effect is null with respect to the good under analysis. Only when the demands for the other goods are not affected by the price change and the income effect is null will this equality hold.

In cases where the modified demands for each good do not depend on the prices of the other goods, preferences are strictly convex and the income effect is null, we've seen that the variation in the extractible surplus provides an extra argument for the choice of the compensating variation, in the case of a price decrease, and the equivalent variation, in the case of a price increase. Besides the usual interpretations these measures are given, they now possess the same interpretation of the Marshallian surplus, in the appropriate cases.

Finally, it is worth mentioning how one the results we obtained in this paper can be used to clarify the discussion about which would be the best among the Hicksian measures. As if there was an objective way of choosing between them, Varian [1992] argues in favor of the equivalent variation, since it would represent the "willingness to pay" of the consumer. However, as we showed above, the value the individual is willing to pay or the value that can be extracted from him is the compensating variation in the case of a price drop and the equivalent variation in the case of a price escalation.

VI. Conclusion

In this paper, we first reviewed the argument that justifies the use of the Marshallian surplus as a measure of welfare, which, as is well known, can only be applied to goods whose consumption plan is not affected by income variations[4]. We also reached the conclusion that the variation in the Marshallian surplus can only be calculated as the trapezoidal area below the Walrasian demand for the good when the demands for the other goods are not affected by the price being altered. Otherwise the demand for at least one of the other goods would vary, and the Marshallian surplus in that market would change accordingly.

The main contribution of this paper is the construction of a new measure of the consumer's surplus, namely the extractible surplus, which extends the Marshallian surplus idea to situation where the assumption of no income effects doesn't hold. The notion of total extractible surplus is no different from the compensating variation as defined by Hicks in *Value and Capital*, namely the value the individual is willing to pay to prevent the market for that good from closing. There are important differences though when the variation in the extractible surplus is considered.

When the price of only one good is allowed to vary, there is an equivalence between the variation in the extractible surplus and the compensating variation, in the case of a price drop, and the equivalent variation, in the case of a price elevation, as long as preferences are strictly convex and the price change of the good under consideration doesn't affect the modified demands for the other goods. If these conditions are satisfied, there is another argument that can be used to justify the extractible surplus notion. Besides sharing, by construction, the interpretation of the Marshallian surplus, the variation in the extractible surplus can be interpreted as the maximum value the individual is willing to pay to avoid a price increase or the maximum value he is willing to pay to keep the price from falling, depending on whether we are in a price rise or price drop context.

[4] When the income effect is not null, there is still the possibility of calculating the variation in the Marshallian surplus as na approximation to the compensating and equivalent variations, as suggested by Willing [1976, 1979]. However, as Takayama [1983] tells us, "... the discrepancy between the Marshallian and the Hicksian measures can be very large (can even been infinitly large)." Moreover, as shown by Hausman [1981], this discrepancy can be very large if we are concerned with the calculation of the deadweight loss.

VII. References

DUPUIT, J. (1844). "Medición de la utilidad de las Obras Públicas" In*: Ensaios sobre la Economia del Bienestar*, ed. K. J. Arrow and T. Scitovsky, pp 319:53, El Trimestre Económico.

HAUSMAN, J. A. (1981). "Exact Consumer's Surplus and Deadweight Loss", *American Economic Review* 71, pp. 662-76.

HICKS, John R. (1939). "Value and Capital", 2ª ed., 1965, Oxford, Clarendon Press, pp. 340.

HICKS, John R. (1941). "The Rehabilitation of Consumers' Surplus", *Review of Economic Studies* 8, pp.108-116.

HICKS, John R. (1942). "Consumer's Surplus and Index-Numbers", *Review of Economic Studies* 9, pp. 126-37.

HICKS, John R. (1943). "The Four Consumer's Surpluses", Review of Economic Studies 11, pp. 31-41.

HICKS, John R. (1945). "The Generalized Theory of Consumer's Surplus", Review of Economic Studies 13, pp.68-74.

HICKS, John R. (1956). "A Revision of Demand Theory", 4ª ed., 1969, Oxford, Clarendon Press, pp 198.

LIMA, Elon Lages (1999). *Análise Real: Volume 1*, "Coleção Matemática Universitária", 4ª ed.,Rio de Janeiro, Instituto de Matemática Pura e Aplicada, 1999, pp. 189.

MARSHALL, Alfred (1890). "Value and Utility" In: *Principles of Economics – An Introdutory Volume*, New York, The Macmillan Company, 8ª ed., pp.124-37.

SAMUELSON, Paul A. (1972). "Constancy of the Marginal Utility of Income" In*: The Collected Scientific Papers of Paul A Samuelson*, edited by Joseph E. Stiglitz, Massachusetts, The Massachusetts Intitute of Technology Press, pp. 88-293.

TAKAYAMA, Akira (1983). "Consumer's Surplus." In: *Analitical Methods in Economics*, Michigan, Michigan University.

VARIAN, Hal R. (1992). "Consumer's Surplus" In: *Microeconomic Analysis*, 3ª ed., New York e London, W. W. Norton & Company, pp. 160-71.

WILLING, R. D. (1976). "Consumer's Surplus without Apology", *American Economic Review* 66, pp. 589-97.

WILLING, R. D. (1979). "Consumer's Surplus Without Apology, Reply." *American Economic Review* 69, pp. 469-74.

VIII. Appendix

Proof 1:

We want to show that the price α such that u(0,M)=u(1,M-α) is the only solution to equation (2) in the main text.

In order for α to be a candidate for maximum of this equation, it is necessary that the optimal choice that emerges from the utility maximization problem subject to the budget restriction when the price of good 1 is equal to α specifies a quantity of good 1 greater than or equal to 1. In particular, if we can show that (1,M-α) is na optimal choice when the price of good 1 is α, this condition will be satisfied.

We will split up the proof in two parts. In the first part we will show that the bundles (0,M) e (1,M-α) are the two only possible solutions to the utility maximization problem when the price of good 1 is α. Then we will show that α is the unique price of good 1 which satisfies equation (2).

Part 1: In order to show that (0,M) and (1,M-α) are the only two solutions to the maximization problem when the price of good 1 is α, we need to show that the budget set and the set of all bundles which are at least as good but not equal to the bundles (0,M) and (1,M-α) are disjoint when the price of good 1 is α.

Notice that the bundle (1,M-α) belongs to the boudary of the budget set when the price of good 1 is α, since when the individual consumes 1 unit of good 1 at price α, the maximum of good 2 he can buy is M-α. We already know that (0,M) belongs to the budget line whatever the price of good 1. Therefore, when the price of good 1 is α, the bundles (0,M) e (1,M-α), between which the individual is indifferent, are also on the budget line. Therefore, the budget line may be represented as the line going through these two bundles, and the budget set is composed of all the points on or below this line.

Now we need the following result:

Result A1: If the condition 8 in the main text is satisfied, then the three distinct and indifferent bundles cannot all be on the same line.

Proof: Suppose, by contradiction, that there are three bundles x', x'' and x''' in X such that x'~ x''~ x''', where x'$\neq$x'', x'$\neq$x''' and x''$\neq$x''', and that these three bundles can be all on some line at the same time. Since the bundles are distinct, it is not possible for two of them to specify the same quantity of good 1, for in this case the quantity of good 2 would have to be different, and then, by strong monotonicity, one of them will be strictly preferred to the other, which is a contradiction. Let's take the bundles with the largest and the smallest quantities of good 1. Without loss of generality, let them be x' and x'', respectively. Since the three bundles can be placed on the same line, we have that x''' can be written as x'''=β x'+(1-β)x'' onde $\beta \in (0,1)$. In this case, assumption 7 implies x'''$\succ$x'~ x'', which is a contradiction. ■

We are now able to show the following result.

Result A.2: If conditions 5 and 7 are satisfied, then the set of all bundles at least as good but not equal to (0,M) e (1,M-α) must lie above the line that passes through these two bundles.

Proof: Suppose, by contradiction, that there exists a bundle at least as good as (0,M) and (1,M-α), which we denote by (k_1,k_2), where (k_1,k_2)$\neq$(0,M) and (k_1,k_2)$\neq$(1,M-α), and that this bundle is located on or below the line that goes through (0,M) and (1,M-α). We have to consider three possible cases: k_1=0, k_1=1 or k_1>1, where $k_1 \in$ N.

<u>Case 1</u>: k_1=0

We know that (k_1,k_2) is on or the line which goes through bundles (0,M) and (k_1,k_2)$\neq$(0,M). Therefore, k_2<M. Since preferences are strongly monotonic, this implies (0,M) $\succ$(k_1,k_2), which is a contradiction.

<u>Case 2</u>: k_1=1

Similarly to the previous case, this possibility can only happen when k_2<M-α, which implies (1,M-α)$\succ$(k_1,k_2), a contradiction.

<u>Case 3</u>: k_1>1, where $k_1 \in$ N

Let (λ_1, λ_2) be a bundle such that (λ_1, λ_2) ~(0,M), where $(\lambda_1, \lambda_2) \in X$.

Take $k_1=\lambda_1$, implying $\lambda_1>1$. Then necessarily $\lambda_2>k_2$, otherwise it couldn't be true that (λ_1, λ_2) ~(0,M).

Now suppose, by contradiction, that (λ_1, λ_2) ~(0,M), $k_1=\lambda_1$ and $\lambda_2 \leq k_2$. In this case, $(\lambda_1, \lambda_2) \leq (k_1,k_2)$. Since (k_1,k_2) is on or below the line which goes through points (0,M) and (1,M-α), we have that (λ_1, λ_2) is also on or below this line.

<u>Sub case 3.1</u>: (λ_1, λ_2) is below the line that goes through (0,M) e (1,M-α).

In this sub case, the line which connects points (λ_1, λ_2) and (0,M) lies below the bundle (1,M-α). Hence, if we take a bundle x''' such that x_1'''=1 and x'''= β(0,M)+(1-β)(λ_1, λ_2) (this is possible because $\lambda_1=k_1>1$), the quantity of good 2, x_2''', will be smaller than M-α, and strong monotonicity implies (1,M-α)$\succ$(1,x_2''') (=(x_1''',x_2''')). But we know that (x_1''',x_2''')$\succ$(0,M) by assumption 7. Therefore, by transitivity, (1,M-α)$\succ$(0,M), and we got our contradiction.

<u>Sub case 3.2</u>: (λ_1, λ_2) is on the line that goes through (0,M) e (1,M-α).

In this sub case, result A.1 is being violated (and consequently condition 7), since (λ_1, λ_2) ~(0,M) ~(1,M-α) and these are three distinct bundles.

Therefore, if (λ_1, λ_2)~(0,M) and $k_1=\lambda_1$, then $\lambda_2 > k_2$ and strong monotonicity implies $(\lambda_1, \lambda_2) \succ (k_1,k_2)$. Since (λ_1, λ_2) ~(0,M), transitivity tells us that (0,M)$\succ$(k_1,k_2), which is absurd, since the original assumption is that (k_1,k_2)$\succsim$(0,M). ■

We go back now to the main proof. We've shown that the budget set can be represented by the bundles which are positioned on or below the budget line which goes through (0,M) and (1,M-α). Result A.2 then tells us that the set of all bundles at least as good as these two bundles is above this line. Therefore the set of bundles at least as good as (0,M) and (1,M-α) and the set of all bundles that satisfy the budget restriction are disjoint if we exclude these two bundles. In other words, bundles which are preferred to or indifferent to (0,M) and (1,M-α) are not in the budget set. Therefore (0,M) and (1,M-α) are the only two solutions to the maximization problem (1) when the price of good 1 is α. Condition 8 guarantees that the individual buys the first unit of good one at this price.

Parte II: We want to show that α is the only solution to equation (2).

Suppose, by contradiction, that there exists a price $p_1' \neq \alpha$ such that p_1' is a solution to (2). Then $p_1' \geq \alpha$. Since $p_1' \neq \alpha$, we have that $p_1' > \alpha$. If p_1' is a solution to (2), then there exists a bundle (δ_1, δ_2) with $\delta_1 \geq 1$ and such that this bundle is a solution to the utility maximixation problem when the price of good 1 is p_1'.

We have then two cases to consider: $\delta_1 = 1$ and $\delta_1 > 1$.

<u>Case 1:</u> $\delta_1 = 1$

Since (0,M) always belongs to the budget line, independently of the price of good 1,in order for (δ_1, δ_2) to be na optimal choice it is necessary that

$$(\delta_1, \delta_2) \succsim (0,M) \qquad \text{(A.1)}$$

Since $p_1' > \alpha$, (δ_1, δ_2) is located below the line that connects (0,M) and $(1, M-\alpha)$. But for this to be true when $\delta_1 = 1$, we need $\delta_2 < M-\alpha$ to hold. By strong monotonicity then, it must be the case that $(1, M-\alpha) \succ (\delta_1, \delta_2)$. Since $(1, M-\alpha) \sim (0,M)$, transitivity implies $(0,M) \succ (\delta_1, \delta_2)$. But this violates condition (A.1).

<u>Case 2</u>: $\delta_1 > 1$

For (δ_1, δ_2) to be an optimal choice, we need condition (A.1) to be satisfied, but it cannot happen that $(\delta_1, \delta_2) \succ (0,M)$, for in this case there would be a bundle indifferent to (0,M) at which the slope of the line connecting the two bundles was greater. Therefore p_1' wouldn't be a solution to (2). We have then that $(\delta_1, \delta_2) \sim (0,M)$. If $\delta_1 > 1$, there exists β such that $(1, M-\alpha) = \beta (0,M) + (1-\beta)(\delta_1, \delta_2)$. By condition 9, we would have then $(1, M-\alpha) \succ (0,M)$, which is a contradiction.

Therefore, assuming there is a $p_1' \neq \alpha$ such that p_1' is a solution to (2), we reach a contradiction in the two possible cases. This means that α is the only solution to (2), and thus $\alpha = \rho_1(1)$.

Combining parts 1 and 2, we conclude that the slope of the line joining bundles (0,M) e (1,M-α) determines $\rho_1(1)$. Therefore $\rho_1(1)$ is the solution to the equation

$$u(0,M)=u(1,M-\rho_1(1)) \blacksquare$$

Proof 2:

If wheat flour is sold in 1 kg packs, the set in which x_1 is defined is IN. If the consumer can buy flour in ½ kg packs, then $x_1 \in IN/2$. For an interval 1/n, where $n \in IN$, the set where x_1 is defined is $IN/2^{n-1}$. Let $P^n = IN/2^{n-1}$ be the set where x_1 is defined when flour can be sold in intervals of 1/n grams. If it is possible to buy flour in any quantity, then $x_1 \in IR$.

The proof will be divided in parts.In part I we will show that, when $n \to \infty$, the set of indifferent points gets close to a curve and that for $x_1 \in IR$ these points will form a curve. In part II, we will show that the indifference curves of the individual when he can buy flour in any quantity is differentiable. Then, at any point x_1, the maximum price he is willing to pay for the consumption of a little bit more of flour is precisely the derivative of the indiffernce curve in that point.Therefore, in each point $x_1 \in IR$, the modified demand will be given by the derivative of the indifference curve. Assuming these curves are differentiable at all $x_1 \in IR$, then the modified demand will be a curve. Finally, we will show that the sum of the areas of all rectangle below the modified demand curve approximates the integral when $n \to \infty$. If the modified demand is a curve, then this are will be the integral.

Part I: When $n \to \infty$, the set of all indifferent point gets close to becoming a curve. For $x_1 \in IR$, these sets do form a curve.

Proof: We will divide this proof in several steps..

Step 1: $P^n \subset P^{n+1}$

By definition, $P^n = IN/2^{n-1}$ and $P^{n+1} = IN/2^n$. To show that $P^n \subset P^{n+1}$, we need to show that each element of P^n is in P^{n+1}. Let $m \in IN$ be an arbitrary element. Then

$m/2^{n-1}$ is na arbitrary element of P^n. We can write $m/2^{n-1}$ in the form $2m/2^n$. Since $m \in IN$, then $2m \in IN$.

Let k=2m, where $k \in IN$. Then $\frac{k}{2^n} \in P^{n+1}$.

<u>Step 2:</u> $\bigcup_{n=1}^{\infty} P^n$ is dense in IR_+.

Proof: To show that $\bigcup_{n=1}^{\infty} P^n$ is dense in IR_+, we need to show that every element $\alpha \in IR_+$ is adherent to $\bigcup_{n=1}^{\infty} P^n$. We say that α is adherent to the set $\bigcup_{n=1}^{\infty} P^n \subset IR_+$ when α is the limit to some sequence of points $x_1 \in \bigcup_{n=1}^{\infty} P^n$. If $\alpha \in \bigcup_{n=1}^{\infty} P^n$, then necessarily α is adherent to $\bigcup_{n=1}^{\infty} P^n$: we just have to take the sequence $x_1 = \alpha$. It remains to show that $\alpha \in IR \setminus \bigcup_{n=1}^{\infty} P^n$ is adherent to $\bigcup_{n=1}^{\infty} P^n$.

If $\alpha \in IR \setminus \bigcup_{n=1}^{\infty} P^n$, then $\alpha \neq \frac{m}{2^{n-1}}$ $\forall m, n \in IN$. Let's show that there exists a sequence $(x_h)_{h \in IR}$ where $x_h \in \bigcup_{n=1}^{\infty} P^n$ and $\lim x_h = \alpha$. Let $n_0 \in IN$ be such that $n_0 < \alpha < n_0 + 1$. Let's divide the interval $(n_0, n_0 + 1)$ where α lies in two intervals of the same size: $(n_0, n_0 + 1/2)$ and $(n_0 + 1/2, n_0 + 1)$. Let $x_1 = n_0 + 1/2$ be the first element of the sequence. Notice that $x_h \in \bigcup_{n=1}^{\infty} P^n$, since $\frac{2n_0 + 1}{2} \in P^2$. Notice also that $\alpha \neq x_1$, since $\alpha \notin \bigcup_{n=1}^{\infty} P^n$. No w we choose the interval which contains α. Without loss of generality, imagine that $\alpha \in (n_0 + 1/2, n_0 + 1)$. Let's divide this interval in two other intervals with the same size, $(n_0 + 1/2, n_0 + 3/4)$ and $(n_0 + 3/4, n_0 + 1)$, and let's take $x_2 = n_0 + 3/4$ as the second element of our sequence. We can assume, without loss of generality, that $\alpha \in (n_0 + 1/2, n_0 + 3/4)$. In this case,

$x_3 = n_0 + 5/8\left(= \dfrac{n_0 + 1/2 + (n_0 + 3/4)}{2}\right)$. If $\alpha \in (n_0 + 3/4, n_0 + 1)$, then $x_4 = n_0 + 7/8$ and so on. We built a sequence where, for each x_h where $h \in IN$, x_h and α belong to an interval of size $1/2^h$. Therefore $|x_h - \alpha| < 1/2^h \quad \forall h \in IN$. Hence the larger h the closer x_h to α, and hence $\lim_{h\to\infty}(x_h) = \alpha$. Since α is arbitrary, IR_+ is the set of points of sdherence of $\bigcup_{n=1}^{\infty} P^n$. Therefore $IR_+ = \overline{\bigcup_{n=1}^{\infty} P^n}$.

Step 3: Since the set $\bigcup_{n=1}^{\infty} P^n$ is dense in IR for each $\alpha \in IR_+$, there exists a decreasing sequence $(t_i)_{i\in IN}$ such that $t_i \in \bigcup_{n=1}^{\infty} P^n$ and $t_i \to \alpha$ and there exists an increasing sequence $(q_i)_{i\in IN}$ such that $q_i \in \bigcup_{n=1}^{\infty} P^n$ and $q_i \to \alpha$.

Let's show now that there exists a decreasing sequence. The proof for an increasing sequence is analogous.

Case 1: If $\alpha \in \bigcup_{n=1}^{\infty} P^n$, then $a = \dfrac{k}{2^{m_0}}$ for some $m_0 \in IN$. In this case, the sequence $t_i = \dfrac{k}{2^{m_0}} + \dfrac{1}{2^i}$ is decreasing and converges to α.

Case 2: If $\alpha \in IR \setminus \bigcup_{n=1}^{\infty} P^n$, consider the sequence $t_1 = n_0 + 1$, where n_0 is the natural number which immediately precedes α. If $\alpha \in \left(n_0, \dfrac{2n_0 + 1}{2}\right)$, take $x_2 = \dfrac{2n_0 + 1}{2}$ and, if $\alpha \in \left(\dfrac{2n_0 + 1}{2}, n_0 + 1\right)$, take $t_2 = n_0 + 1$. Suppose, without loss of generality, that $\alpha \in \left(n_0, \dfrac{2n_0 + 1}{2}\right)$. If $\alpha \in \left(n_0, \dfrac{4n_0 + 1}{2}\right)$, take $t_3 = 4n_0 + 1$ and, if $\alpha \in \left(\dfrac{4n_0 + 1}{4}, \dfrac{2n_0 + 1}{2}\right)$, take $t_3 = \dfrac{2n_0 + 1}{2}$. Proceeding in this fashion, we construct a decreasing sequence where, given $n \in IN$, there exists $t_i \in \bigcup_{n=1}^{\infty} P^n$ (due to the density of

$\bigcup_{n=1}^{\infty} P^n$) such that $|t_i - \alpha| < \frac{1}{2^{i-1}}$. Thus, when *i* increases this sequence converges to α when $i \to \infty$.

Step 4: $f : IR_+ \to IR_+$ is continuous.

Let f(.) be a function that associates to each $x_1 \in P^n$ a $x_2 \in IR$ in such a way that the bundle $(x_1, f(x_1))$ generates the same level of utility for the individual $\forall x_1 \in \bigcup_{n=1}^{\infty} P^n$.

We know that a function f(.) is continuous at $x \in IR$ if, and only if, for every sequence $(a_i)_{i \in IN} \subset IR$ such that $a_i \to x$,

$$\lim_{t_i \to x^+} f(t_i) = \lim_{q_i \to x^-} f(q_i) = f(x)$$

Let $\alpha \in IR_+$. Consider a decreasing sequence such that $t_i \to \alpha$ and $t_i \in \bigcup_{n=1}^{\infty} P^n$ (by step 3, such a sequence exists). Consider also an increasing sequence such that $q_i \to \alpha$. We have that $\forall m, k \in IN$

$$q_m \leq q_{m+1} \leq \alpha \leq t_{k+1} \leq t_k$$

Since $f : \bigcup_{n=1}^{\infty} P^n \to IR$ is decreasing, we have that

$$f(q_m) \geq f(q_{m+1}) \geq f(t_{k+1}) \geq f(t_k) \qquad \text{(A.2.)}$$

Hence $f(q_m)$ cannot be smaller that $f(t_{k+1})$ $\forall m \in IN$ and $(f(q_m))_{m \in IN}$ is a decreasing and bounded seuqnece. By the Bolzano-Weierestrass theorem[5], this sequence is convergent.

Similarly, $(f(t_k))_{k \in IN}$ cannot be greater than $f(q_m)$, which means that $f(t_{k+1})$ is an increasing and bounded sequence, and therefore convergent.

Since (A.2) is true $\forall m, k \in IN$, we have that

$$\lim_{i \to \infty} f(q_i) \geq \lim_{i \to \infty} f(t_i) \qquad \text{(A.3.)}$$

[5] See, for instance, Lima [1999].

Let $a = \lim_{i\to\infty} f(q_i)$ and $b = \lim_{i\to\infty} f(t_i)$. According to (A.3.), we have $a \geq b$. We want to show that a=b.

Suppose, by contradiction, that $a \neq b \Rightarrow a > b$. Let $m_0, k_0 \in IN$ be sufficiently large so that the line segment

$$\lambda(q_{m_0}, f(q_{m_0})) + (1-\lambda)(t_{k_0}, f(t_{k_0}))$$

is located below the point (α,a) and above the point (α,b). This is possible because a>b. Since $(q_m, f(q_m)) \to (\alpha, a)$, there exists a $m_1 > m_0$ such that the point $(q_{m_1}, f(q_{m_1}))$ is above the line segment $\lambda(q_{m_0}, f(q_{m_0})) + (1-\lambda)(t_{k_0}, f(t_{k_0}))$. However, this is a contradiction, for f(.) is strictly convex. Therefore

$$\lim_{i\to\infty} f(q_i) = a = b = \lim_{i\to\infty} f(t_i)$$

We have to consider two cases now.

Case 1: If $\alpha \in \bigcup_{n=1}^{\infty} P^n$ then f(α)=a.

Suppose, by contradiction, that f(α)≠ a.

Sub case 1: f(α)>a

Since $a = \lim_{i\to\infty} f(q_i)$, we have that $f(q_i) \geq a$ for all $i \in IN$. If f(α)>a, there is an $m_0 \in IN$ sufficiently large such that f(α)> $f(q_{m_0}) \geq a$. Since f(.) is decreasing, we would have then $\alpha \leq q_{m_0}$, which is a contradiction, for $(q_i)_{i\in IN}$ is na increasing sequence such that $q_i \to \alpha$.

Sub case 2: f(α)<a

Since $a = \lim_{i\to\infty} f(t_i)$, we have that $f(t_i) \leq a$ for all $i \in IN$. If f(α)<a, then there is a $k_0 \in IN$ sufficiently large such that f(α)< $f(t_{k_0}) \leq a$. Since f(.) is decreasing, we would have then $\alpha \geq t_{k_0}$, a contradiction, for $(t_i)_{i\in IN}$ is a decreasing sequence such that $t_i \to \alpha$.

Therefore we conclude that f(.) is continuous for all $x_h \in \bigcup_{n=1}^{\infty} P^n$.

<u>Case 2:</u> $\alpha \in IR \setminus \bigcup_{n=1}^{\infty} P^n$

In this case the individual can buy flour in any given quantity and we wish to show that the function is continuous at these points. Suppose, by contradiction, that $f(\alpha) \neq a$.

Since $IR \supset \bigcup_{n=1}^{\infty} P^n$, the same argument of convergent sequences used in sub cases 1 and 2 of the previous case can be applied here to reach a contradiction.

Part II: The modified demand is a curve when we allow the consumer to buy wheat flour in any amount. In this case, the extractible surplus is the integral of the modified demand over $\left[\underline{p}_1, \overline{p}_1\right]$.

<u>Step 1:</u> The slope of the line that connects points $(x_1, f(x_1))$ and $(x_1 + 1/n, f(x_1 + 1/n))$, where $x_1 \in IR$, is the derivative of the function f(.) at x_1 when $n \to \infty$ if $f: IR \to IR$ is differentiable.

If f(.) is differentiable at $x_1 \in IR$, then

$$f'(x_1) = \lim_{h \to 0} \frac{f(x_1 + h) - f(x_1)}{h} \qquad \text{(A.4.)}$$

In our case, we are calculating

$$f'(x_1) = \lim_{n \to \infty} \frac{f(x_1 + 1/n) - f(x_1)}{1/n}$$

If we substitute h for 1/n we get exactly equation (A.4).

Therefore the modified demand that associates to each x_1 the slope of the line joining $(x_1, f(x_1))$ and $(x_1 + 1/n, f(x_1 + 1/n))$ when $n \to \infty$ is a curve that associates to each x_1 the derivative of the indifference curve with utility level $u(\overline{x}_1, M - \overline{p}_1\overline{x}_1)$ at point x_1.

<u>Step 2:</u> When $n \to \infty$, the sum of the areas of all rectangles with height given by the difference between the price on the modified demand curve and $\underline{p}_1$ is the integral of the modified demand between $\overline{p}_1$ e $\underline{p}_1$.

Notice that, as $n \to \infty$ the sum of all the areas of these rectangles approaches the total area below the modified demand curve. A standard argument shows then that, when the individual can buy wheat flour in continuous quantities (the modified demand is a curve in this case), the sum of the areas of the rectangles below this curve is precisely the integral of the modified demand over $\left[\underline{p}_1, \overline{p}_1\right]$. ■

The ECO/UnB Working Paper Series

The Department of Economics of the University of Brasilia publishes its Working Papers Series since April 1972. On August 30, 2002 the series was renewed with the on-line publication of the papers. All Working Papers may be freely downloaded from the Department site: http://www.unb.br/ih/eco.

Working papers published since August 2002:

231 Posse de escravos e estrutura da riqueza no agreste e sertão de Pernambuco: 1777-1887. Flávio Rabelo Versiani and José Raimundo O. Vergolino, 30 August 2002, 29p.

232 On the natural rates of unemployment and interest: the Robertson connection. Mauro Boianovsky and John R. Presley, 30 August 2002, 34p.

233 Contas Nacionais e o meio ambiente: reflexões em torno de uma abordagem para o Brasil. Charles C. Mueller, 30 August 2002, 25p.

234 Economics of air pollution: hedonic price model and smell consequences of sewage treatment plants in urban areas. Sérgio A. Batalhone, Jorge M. Nogueira and Bernardo P. M. Mueller, 30 August 2002, 25p.

235 The Brazilian depression of the 80s and 90s. Mirta Bugarin, Roberto de G. Ellery Jr., Victor Gomes and Arilton Teixeira, 30 August 2002, 30p.

236 Informal employment in Brazil – A choice at the top and segmentation at the bottom: a quantile regression approach. Maria Tannuri-Pianto and Donald M. Pianto, 30 August 2002, 23p.

237 False contagion and false convergence clubs in stochastic growth theory. Stephen de Castro and Flávio Gonçalves, 30 August 2002, 20p.

238 Spot and contract markets in the Brazilian wholesale energy market. Paulo C. Coutinho and André Rossi de Oliveira, 30 August 2002, 19p.

239 Tributação da renda e do consumo no Brasil: uma abordagem macroeconômica. Valter Borges de Araújo Neto e Maria da C. S. de Sousa, 30 August 2002, 31p.

240 Vote splitting, reelection and electoral control: towards a unified model. Maurício S. Bugarin. 30 August 2002, 26p.

241 Shadow-prices in payment systems. Rodrigo Peñaloza, 6 September 2002, 31p.

242 Welfare implications of the Brazilian social security system. Roberto de G. Ellery Jr. and Mirta N. S. Bugarin, 13 September 2002, 28p.

243 Os agentes econômicos em processo de integração regional – Inferências para avaliar os efeitos da ALCA. Renato Baumann and Francisco Galrão Carneiro, 13 September 2002, 29p.

244 Leading by example: a simple evolutionary approach. André Rossi de Oliveira and João R. O. de Faria, 20 September 2002, 25p.

245 The role of institutions in sustainable development. Bernardo Mueller and Charles Mueller, 20 September 2002, 25p.

246 Incentivos em consórcios intermunicipais de saúde: uma abordagem de teoria dos contratos. Luciana Teixeira, Maria Cristina MacDowell and Mauricio Bugarin, 27 September 2002, 19p.

247 Liquidity constraints and the behavior of aggregate consumption over the Brazilian business cycle. Mirta Bugarin and Roberto de G. Ellery Jr, 27 September 2002, 19p.

248 Pricing water and sewage services in urban areas: Evidences of low level equilibrium in a developing economy. Ricardo Coelho de Faria, Jorge M. Nogueira and Bernardo Mueller, 4 October 2002.

249 Wrong incentives for growth in the transition from modern slavery to labor markets: Babylon before, Babylon after. Stephen de Castro, 4 October 2002, 23p.

250 Vintage capital, distortions and development. Samuel Pessoa and Rafael Rob, 11 October 2002, 40p.

251 Consórcios intermunicipais de saúde: uma análise à luz da teoria dos jogos. Luciana Teixeira, Maria Cristina MacDowell and Mauricio Bugarin, 11 October 2002, 30p.

252 Preços de escravos em Pernambuco no século XIX. Flávio R. Versiani and José Raimundo O. Vergolino, 18 October 2002, 20p.

253 A model of capital accumulation and rent seeking. Paulo Barelli and Samuel de Abreu Pessoa, 18 October 2002, 47p.

254 Anchors away: the cost and benefits of Brazil's devaluation. Edmund Amann and Werner Baer, 25 October 2002, 20p.

255 Um seguro agrícola "eficiente". Aércio S. Cunha, 25 October 2002, 57p.

256 Campaign contributions with swing voters. Manfred Dix and Rudy Santore, 1 November 2002, 15p.

257 Incentivos para os administradores de empresas estatais: O papel dos dividendos mínimos obrigatórios e o desenho ótimo de salários. André Luís G. Carcia and Maurício Bugarin, 1 November 2002, 28p.

258 Impostos e a História. Aércio S. Cunha, 8 November 2002, 12p.

259 Determinantes do endividamento dos estados brasileiros: Uma análise de dados de painel. Isabela Fonte Boa Rosa Silva e Maria da Conceição Sampaio de Sousa, 8 November 2002, 27p.

260 Technology adoption: On the nonequivalence of tariffs and quotas. Arilton Teixeira, 15 November 2002, 25p.

261 Constitutional regimes, growth and stagnation in the Brazilian economy: 1947-1999. Marco Antônio Campos Martins, 15 November 2002, 39p.

262 Price caps and electoral cycles. César Mattos, 22 November 2002, 16p.

263 Os pobres que levantem a mão (mas será que são mesmo pobres?) - Uma tentativa de validar o cadastro único. Carlos Alberto Ramos and Ricardo Santana, 29 November 2002, 100p.

264 Relative earnings of immigrants and natives under changes in the US wage structure, 1970-1990: A quantile regression approach. Maria Tannuri-Pianto, 29 November 2002, 40p.

265 Bidding strategies in the Brazilian Treasury auctions. Anderson Caputo Silva, 6 December 2002, 34p.

266 Crises cambiais e ataques especulativos no Brasil. Mauro Costa Miranda, 13 December 2002, 26p.

267 Poverty and environment degradation: the Kuznets environmental curve for the Brazilian case. Fabio G. e Barros, Augusto F. Mendonça and Jorge M. Nogueira, 20 December 2002, 27p.

268 On shadow-prices of banks in real-time gross settlement systems. Rodrigo Peñaloza, 20 December 2002, 31 p.

269 A characterization of renegotiation-proof contracts via random fixed points in Banach spaces. Rodrigo Peñaloza, 20 December 2002, 9 p.

270 Existence of time-invariant settlements in FEDWIRE-like payment systems. Rodrigo Peñaloza, 27 December 2002, 13p.

271 Principal-Agent problem with continuum of constraints: the infinite dimensional approach.

Rodrigo Peñaloza, 27 December 2002, 43p.

272 Structural analysis of multiple-unit auctions: recovering bidders' valuations in auctions with dominant bidders. Anderson Caputo Silva, January 3, 2003, 18 p.

273 Financiamento público de campanhas eleitorais: efeitos sobre bem-estar social e representação partidária no Legislativo. Adriana C. Portugal and Maurício S. Bugarin, January 10, 2003, 25p.

274 Wicksell on technical change, real wages and employment. Mauro Boianovsky and Harald Hagemann, January 17, 2003, 28p.

275 Quão pobres são os pobres. Brasil: 1992-2001. Carlos Alberto Ramos and Ricardo Santana, January 24, 2003, 20p.

276 Dois anos da Lei de Responsabilidade Fiscal do Brasil: uma avaliação dos resultados à luz do modelo do fundo comum. Selene Peres Peres Nunes and Ricardo da Costa Nunes, January 31, 2003, 45p.

277 Políticas de geração de emprego e renda: Justificativas teóricas, contexto histórico e experiência brasileira. Carlos Alberto Ramos, February 7, 2003, 28p.

278 Long run implications of the Brazilian capital stock and income estimates. Victor Gomes, Mirta N. S. Bugarin and Roberto Ellery Jr., February 14, 2003, 25p.

279 Taxation on intergenerational bequest and redistribution of wealth in a class-setting. Mauro Baranzini, Sheila Oliveira Benjuino and Joanílio Rodolpho Teixeira, February 21, 2003, 23p.

280 On portfolio management. Paulo Coutinho and Benjamin Miranda Tabak, February 28, 2003, 21p.

281 Decentralized portfolio management. Paulo Coutinho and Benjamin Miranda Tabak, March 7, 2003, 22p.

282 The IS-LM model and the liquidity trap concept: From Hicks to Krugman. Mauro Boianovsky, March 14, 2003, 43p.

283 À espera da reforma orçamentária: um mecanismo temporário para redução de gastos públicos. Carla G. Protásio, Maurício S. Bugarin and Mirta N. S. Bugarin, March 21, 2003, 31p.

284 A dívida pública interna e sua trajetória recente. Flávio Rabelo Versiani, March 28, 2003, 16p.

285 Investment and capital accumulation in Brazil from 1970 to 2000: a neoclassical view. Roberto Ellery Jr., Mirta N. S. Bugarin, Victor Gomes and Arilton Teixeira, April 4, 2003, 20p.

286 Setting the right expectations: a note on Carl Walsh's market discipline paper. Fábia Carvalho and Maurício Bugarin, April 11, 2003, 33p.

287 The fifth consumer's surplus: An extension of the concept of Marshallian surplus to preferences with non-null income effects. Cassia Helena Marchon and André Luís Rossi de Oliveira, April 18, 2003, 32p.

Forthcoming working papers:
(Subject to change)

288 Volume, composição e sustentabilidade da dívida pública de liquidez brasileira no período 1994-2002. Fernando de Aquino Fonseca Neto and Joanílio Rodolpho Teixeira, April 25, 2003, 35p.

289 A tarifa de acesso na indústria de telecomunicação quando a hipótese de monopólio natural é quebrada. Rafael de Melo Silveira and André Rossi de Oliveira, May 2, 2003, 20p.